Exercise 9-2

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Exercise On the inner product space in Exercise 9-1, for the functions $\boldsymbol{f}_1(x) = x$ and $\boldsymbol{f}_2(x) = x^2$ calculate the values of $\|\boldsymbol{f}_1 + \boldsymbol{f}_2\|$ and $\|\boldsymbol{f}_1\| + \|\boldsymbol{f}_2\|$ and compare them.

Solution

$$
\begin{aligned}
\|\boldsymbol{f}_1 + \boldsymbol{f}_2\| &= \sqrt{(\boldsymbol{f}_1 + \boldsymbol{f}_2, \boldsymbol{f}_1 + \boldsymbol{f}_2)} \\
&= \sqrt{\int_0^1 \{(\boldsymbol{f}_1 + \boldsymbol{f}_2)(x)\}\{(\boldsymbol{f}_1 + \boldsymbol{f}_2)(x)\}\mathrm{d}x} \\
&= \sqrt{\int_0^1 \{f_1(x) + f_2(x)\}\{f_1(x) + f_2(x)\}\mathrm{d}x} \\
&= \sqrt{\int_0^1 (x + x^2)(x + x^2)\mathrm{d}x} \\
&= \sqrt{\int_0^1 x^2 + 2x^3 + x^4 \mathrm{d}x} \\
&= \sqrt{\left[\frac{x^3}{3} + \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1} \\
&= \sqrt{\frac{1}{3} + \frac{1}{2} + \frac{1}{5}} \\
&= \sqrt{\frac{10 + 15 + 6}{30}} \\
&= \sqrt{\frac{31}{30}} \\
\|\boldsymbol{f}_1\| + \|\boldsymbol{f}_2\| &= \sqrt{(\boldsymbol{f}_1, \boldsymbol{f}_1)} + \sqrt{(\boldsymbol{f}_2, \boldsymbol{f}_2)} \\
&= \sqrt{\int_0^1 x^2 \mathrm{d}x} + \sqrt{\int_0^1 x^4 \mathrm{d}x}
\end{aligned}
$$

$$\begin{aligned}
&= \sqrt{\left[\frac{x^3}{3}\right]_0^1} + \sqrt{\left[\frac{x^5}{5}\right]_0^1} \\
&= \sqrt{\frac{1}{3}} + \sqrt{\frac{1}{5}}
\end{aligned}$$

We compare $\|\boldsymbol{f}_1 + \boldsymbol{f}_2\|^2$ and $(\|\boldsymbol{f}_1\| + \|\boldsymbol{f}_2\|)^2$.

$$\begin{aligned}
(\|\boldsymbol{f}_1\| + \|\boldsymbol{f}_2\|)^2 - \|\boldsymbol{f}_1 + \boldsymbol{f}_2\|^2 &= \left(\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{5}}\right)^2 - \left(\sqrt{\frac{31}{30}}\right)^2 \\
&= \frac{1}{3} + \frac{1}{5} + 2\sqrt{\frac{1}{15}} - \frac{31}{30} \\
&= \frac{10 + 6 - 31}{30} + 2\sqrt{\frac{1}{15}} \\
&= -\frac{15}{30} + 2\sqrt{\frac{1}{15}} \\
&= 2\sqrt{\frac{1}{15}} - \frac{1}{2}
\end{aligned}$$

Here we compare $\left(2\sqrt{\frac{1}{15}}\right)^2$ and $\left(\frac{1}{2}\right)^2$.

$$\begin{aligned}
\left(2\sqrt{\frac{1}{15}}\right)^2 - \left(\frac{1}{2}\right)^2 &= \frac{4}{15} - \frac{1}{4} \\
&= \frac{16 - 15}{60} \\
&= \frac{1}{60} \\
&\geq 0
\end{aligned}$$

So we obtain $2\sqrt{\frac{1}{15}} - \frac{1}{2} \geq 0$ and hence the following inequality.

$$\|\boldsymbol{f}_1\| + \|\boldsymbol{f}_2\| > \|\boldsymbol{f}_1 + \boldsymbol{f}_2\|$$

We have shown that the Triangle inequality holds for the above example.