Exercise 8

Isao Sasano

Exercise 7 Calculate $T_4(x)$.

Solution 1

$$\begin{aligned}
\cos 4\theta &= \cos(3\theta + \theta) \\
&= \cos 3\theta \cos\theta - \sin 3\theta \sin\theta \\
&= (\cos 2\theta \cos\theta - \sin 2\theta \sin\theta)\cos\theta - (\sin 2\theta \cos\theta + \cos 2\theta \sin\theta)\sin\theta \\
&= \{(2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta\}\cos\theta \\
&\quad -\{2\sin\theta\cos\theta\cos\theta + (2\cos^2\theta - 1)\sin\theta)\}\sin\theta \\
&= \{(2\cos^2\theta - 1)\cos^2\theta - 2\sin^2\theta\cos^2\theta\} \\
&\quad -\{2\sin^2\theta\cos^2\theta + (2\cos^2\theta - 1)\sin^2\theta)\} \\
&= \{2\cos^4\theta - \cos^2\theta - 2(1-\cos^2\theta)\cos^2\theta\} \\
&\quad -\{2(1-\cos^2\theta)\cos^2\theta + (2\cos^2\theta - 1)(1-\cos^2\theta))\} \\
&= 2\cos^4\theta - \cos^2\theta - 2\cos^2\theta + 2\cos^4\theta \\
&\quad -\{2\cos^2\theta - 2\cos^4\theta - 2\cos^4\theta + 3\cos^2\theta - 1\} \\
&= 4\cos^4\theta - 3\cos^2\theta - \{-4\cos^4\theta + 5\cos^2\theta - 1\} \\
&= 4\cos^4\theta - 3\cos^2\theta + 4\cos^4\theta - 5\cos^2\theta + 1 \\
&= 8\cos^4\theta - 8\cos^2\theta + 1
\end{aligned}$$

Thus we obtain $T_4(x) = 8x^4 - 8x^2 + 1$.

Solution 2 By applying the recurrence formula of Chebyshev polynomials

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$$

to $T_3(x) = 4x^3 - 3x$ and $T_2(x) = 2x^2 - 1$, we obtain $T_4(x)$ as follows.

$$\begin{aligned}
T_4(x) &= 2xT_3(x) - T_2(x) \\
&= 2x(4x^3 - 3x) - (2x^2 - 1) \\
&= 8x^4 - 6x^2 - 2x^2 + 1 \\
&= 8x^4 - 8x^2 + 1
\end{aligned}$$