A solution for Exercise 6

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Exercise Calculate the Fourier series expansion of the function $f(x) = x^2$ on the range $[-\pi, \pi]$.
Solution Assume the following equation holds. (Note: There are no coefficients a_0, ..., a_n, b_1, ..., b_n that satisfy the equation, but it's ok.)

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \tag{1}$$

Integrate the both sides of the equation (1) on the range $[-\pi, \pi]$.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\mathrm{d}x &= \int_{-\pi}^{\pi} \frac{1}{2}a_0\mathrm{d}x \\ &= a_0\pi\end{aligned}$$

Then we calculate a_0 as follows.

$$\begin{aligned}a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\mathrm{d}x \\ &= \frac{2}{3}\pi^2\end{aligned}$$

Multiply the both sides of the equation (1) by $\cos kx$ and integrate them on the range $[-\pi, \pi]$.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos kx\mathrm{d}x &= a_k\int_{-\pi}^{\pi} \cos^2 kx\mathrm{d}x \\ &= a_k\pi\end{aligned}$$

Then we calculate a_k as follows.

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos kx\mathrm{d}x \\
&= \frac{1}{\pi}\left\{\left[x^2\frac{\sin kx}{k}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\frac{\sin kx}{k}\mathrm{d}x\right\} \\
&= -\frac{2}{\pi k}\int_{-\pi}^{\pi} x\sin kx\mathrm{d}x \qquad \left(\text{since } \left[x^2\frac{\sin kx}{k}\right]_{-\pi}^{\pi} \text{ is } 0\right)
\end{aligned}$$

Here we calculate the integral $\int_{-\pi}^{\pi} x\sin kx\mathrm{d}x$.

$$\begin{aligned}
\int_{-\pi}^{\pi} x\sin kx\mathrm{d}x &= \left[x\frac{-\cos kx}{k}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{-\cos kx}{k}\mathrm{d}x \\
&= -\frac{1}{k}\left[x\cos kx\right]_{-\pi}^{\pi} \qquad \left(\text{since } \int_{-\pi}^{\pi}\frac{-\cos kx}{k}\mathrm{d}x \text{ is } 0\right) \\
&= -\frac{1}{k}(\pi\cos\pi k - (-\pi)\cos(-\pi k)) \\
&= -\frac{1}{k}(\pi\cos\pi k + \pi\cos\pi k) \\
&= -\frac{2\pi}{k}\cos\pi k \\
&= -\frac{2\pi}{k}(-1)^k
\end{aligned}$$

We resume the calculation of a_k.

$$\begin{aligned}
a_k &= -\frac{2}{\pi k}(-\frac{2\pi}{k}(-1)^k) \\
&= \frac{4}{k^2}(-1)^k
\end{aligned}$$

Multiply the both sides of the equation (1) by $\sin kx$ and integrate them on the range $[-\pi, \pi]$.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\sin kx\mathrm{d}x &= b_k\int_{-\pi}^{\pi}\sin^2 kx\mathrm{d}x \\
&= b_k\pi
\end{aligned}$$

Then we calculate b_k as follows.

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx \mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin kx \mathrm{d}x \\ &= 0 \quad (\text{since } x^2 \sin kx \text{ is an odd function}) \end{aligned}$$

So the linear combination that is closest to the function $f(x)$ is

$$\frac{\pi^2}{3} + \sum_{k=1}^{n} \frac{4}{k^2}(-1)^k \cos kx.$$

The Fourier expansion of $f(x) = x^2$ is the limit of the above linear combination as n goes to infinity:

$$\frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2}(-1)^k \cos kx.$$

Supplement We depict the partial summation of this series to the term of $\cos 5x$

$$\frac{\pi^2}{3}+\sum_{k=1}^{5} \frac{4}{k^2}(-1)^k \cos kx = \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \frac{1}{4}\cos 4x - \frac{4}{25}\cos 5x$$

and $f(x) = x^2$ in Fig. 1.

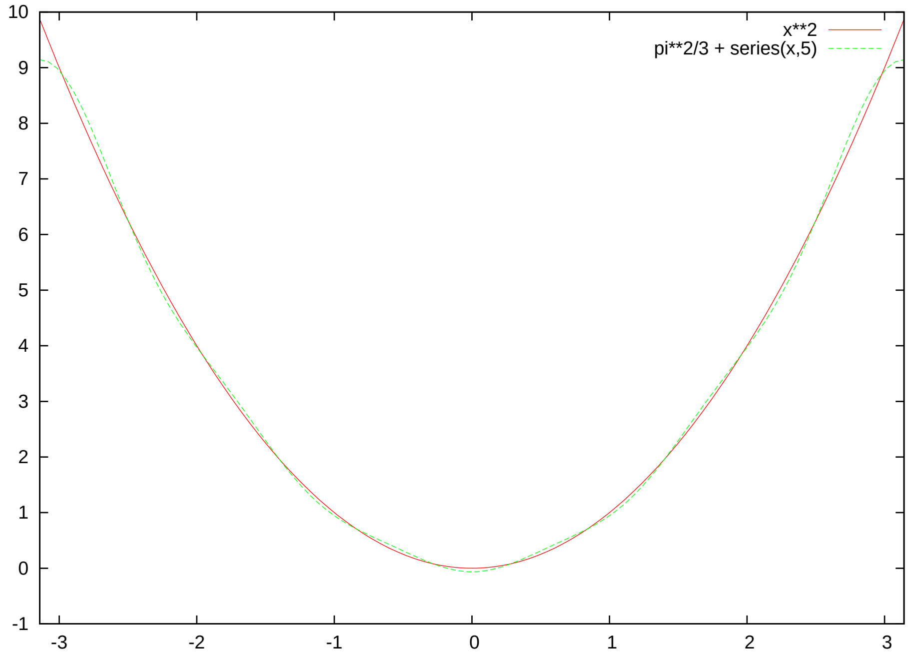

Figure 1: Comparison between the function $f(x) = x^2$ and the partial sum up to the term of $\cos 5x$