Three solutions for Exercise 10-2

Isao Sasano

Exercise Consider the set of real-valued continuous functions on the interval $[-\pi, \pi]$. As we did in Exercise 9-1, we construct an inner product space where the addition, scalar multiplication, and inner product are defined as follows.

$$\begin{aligned} (\boldsymbol{f} + \boldsymbol{g})(x) &= f(x) + g(x) \\ (c\boldsymbol{f})(x) &= c(f(x)) \\ (\boldsymbol{f}, \boldsymbol{g}) &= \int_{-\pi}^{\pi} f(x)g(x)\mathrm{d}x \end{aligned}$$

On this inner product space, approximate a function $f(x) = x^2$ by a linear combination of the functions $e_1(x) = \dfrac{1}{2}$, $e_2(x) = \cos x$, and $e_3(x) = \sin x$ (i.e., $\displaystyle\sum_{k=1}^{3} c_k e_k(x) = c_1 e_1(x) + c_2 e_2(x) + c_3 e_3(x)$ for some c_1, c_2, and c_3). That is, obtain c_1, c_2, c_3 so that $c_1 e_1(x) + c_2 e_2(x) + c_3 e_3(x)$ is closest to $f(x)$. As for the measure of the distance, use (the half of) the square of the norm of the difference of $c_1 e_1(x) + c_2 e_2(x) + c_3 e_3(x)$ and $f(x)$.

$$J = \frac{1}{2} \left\| \sum_{k=1}^{3} c_k \boldsymbol{e}_k - \boldsymbol{f} \right\|^2$$

The norm is defined as follows.

$$\|\boldsymbol{f}\| = \sqrt{(\boldsymbol{f}, \boldsymbol{f})} = \sqrt{\int_{-\pi}^{\pi} f(x)^2 \mathrm{d}x}$$

We show three solutions. One is by solving $\dfrac{\partial J}{\partial c_i} = 0$ for $i = 1, 2, 3$. Another is by considering the special case. The other is by calculating the projection of $\boldsymbol{f}$ on the subspace spanned by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$.

Solution 1 Firstly calculate J as follows.

$$\begin{aligned} J &= \frac{1}{2}\left\|\sum_{k=1}^{3} c_k \boldsymbol{e}_k - \boldsymbol{f}\right\|^2 \\ &= \frac{1}{2}\left(\sum_{k=1}^{3} c_k \boldsymbol{e}_k - \boldsymbol{f}, \sum_{k=1}^{3} c_k \boldsymbol{e}_k - \boldsymbol{f}\right) \\ &= \frac{1}{2}\left\{\left(\sum_{k=1}^{3} c_k \boldsymbol{e}_k, \sum_{k=1}^{3} c_k \boldsymbol{e}_k\right) - 2\left(\boldsymbol{f}, \sum_{k=1}^{3} c_k \boldsymbol{e}_k\right) + \|\boldsymbol{f}\|^2\right\} \\ &= \frac{1}{2}\left\{\sum_{k,l=1}^{3} c_k c_l (\boldsymbol{e}_k, \boldsymbol{e}_l) - 2\sum_{k=1}^{3} c_k (\boldsymbol{f}, \boldsymbol{e}_k) + \|\boldsymbol{f}\|^2\right\} \end{aligned}$$

Partially differentiate this with respect to c_i $(i = 1, 2, 3)$.

$$\begin{aligned} \frac{\partial J}{\partial c_i} &= \frac{\partial}{\partial c_i}\frac{1}{2}\left\{\sum_{k,l=1}^{3} c_k c_l (\boldsymbol{e}_k, \boldsymbol{e}_l) - 2\sum_{k=1}^{3} c_k (\boldsymbol{f}, \boldsymbol{e}_k) + \|\boldsymbol{f}\|^2\right\} \\ &= \frac{1}{2}\left\{\frac{\partial}{\partial c_i}\sum_{k,l=1}^{3} c_k c_l (\boldsymbol{e}_k, \boldsymbol{e}_l) - 2\frac{\partial}{\partial c_i}\sum_{k=1}^{3} c_k (\boldsymbol{f}, \boldsymbol{e}_k)\right\} \\ &= \frac{1}{2}\left\{2\sum_{k=1}^{3} c_k (\boldsymbol{e}_k, \boldsymbol{e}_i) - 2(\boldsymbol{f}, \boldsymbol{e}_i)\right\} \\ &= \sum_{k=1}^{3} c_k (\boldsymbol{e}_k, \boldsymbol{e}_i) - (\boldsymbol{f}, \boldsymbol{e}_i) \end{aligned}$$

By writing $\dfrac{\partial J}{\partial c_i} = 0$ for $i = 1, 2, 3$ in matrix form we obtain

$$\begin{pmatrix} (\boldsymbol{e}_1, \boldsymbol{e}_1) & (\boldsymbol{e}_2, \boldsymbol{e}_1) & (\boldsymbol{e}_3, \boldsymbol{e}_1) \\ (\boldsymbol{e}_1, \boldsymbol{e}_2) & (\boldsymbol{e}_2, \boldsymbol{e}_2) & (\boldsymbol{e}_3, \boldsymbol{e}_2) \\ (\boldsymbol{e}_1, \boldsymbol{e}_3) & (\boldsymbol{e}_2, \boldsymbol{e}_3) & (\boldsymbol{e}_3, \boldsymbol{e}_3) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} (\boldsymbol{f}, \boldsymbol{e}_1) \\ (\boldsymbol{f}, \boldsymbol{e}_2) \\ (\boldsymbol{f}, \boldsymbol{e}_3) \end{pmatrix}$$

Since $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ are orthogonal to each other, we obtain

$$\begin{pmatrix} \|\boldsymbol{e}_1\|^2 & 0 & 0 \\ 0 & \|\boldsymbol{e}_2\|^2 & 0 \\ 0 & 0 & \|\boldsymbol{e}_3\|^2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} (\boldsymbol{f}, \boldsymbol{e}_1) \\ (\boldsymbol{f}, \boldsymbol{e}_2) \\ (\boldsymbol{f}, \boldsymbol{e}_3) \end{pmatrix}$$

Hence we obtain $c_i = \dfrac{(\boldsymbol{f}, \boldsymbol{e}_i)}{\|\boldsymbol{e}_i\|^2}$ for $i = 1, 2, 3$.

$$
\begin{aligned}
(\boldsymbol{f}, \boldsymbol{e}_1) &= \int_{-\pi}^{\pi} \frac{x^2}{2} \mathrm{d}x = \left[\frac{x^3}{6}\right]_{-\pi}^{\pi} = \frac{\pi^3}{3} \\
(\boldsymbol{f}, \boldsymbol{e}_2) &= \int_{-\pi}^{\pi} x^2 \cos x \mathrm{d}x \\
&= \left[x^2 \sin x\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \sin x \mathrm{d}x \\
&= -2 \int_{-\pi}^{\pi} x \sin x \mathrm{d}x \\
&= -2 \left\{ \left[x \frac{\cos x}{-1}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\cos x}{-1} \mathrm{d}x \right\} \\
&= 2 \left[x \cos x\right]_{-\pi}^{\pi} \\
&= 2 \left\{ \pi \cos \pi - (-\pi) \cos(-\pi) \right\} \\
&= 2\{2\pi \cos \pi\} \\
&= 4\pi \cos \pi \\
&= -4\pi \\
(\boldsymbol{f}, \boldsymbol{e}_3) &= \int_{-\pi}^{\pi} x^2 \sin x \mathrm{d}x = 0 \\
(\boldsymbol{e}_1, \boldsymbol{e}_1) &= \int_{-\pi}^{\pi} \frac{1}{4} \mathrm{d}x = \left[\frac{x}{4}\right]_{-\pi}^{\pi} = \frac{\pi}{2} \\
(\boldsymbol{e}_2, \boldsymbol{e}_2) &= \int_{-\pi}^{\pi} \cos^2 x \mathrm{d}x = \pi
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
c_1 &= \frac{\frac{\pi^3}{3}}{\frac{\pi}{2}} = \frac{2}{3}\pi^2 \\
c_2 &= \frac{-4\pi}{\pi} = -4 \\
c_3 &= 0
\end{aligned}
$$

Thus the linear combination is obtained as follows.

$$\frac{2}{3}\pi^2 \boldsymbol{e}_1 - 4\boldsymbol{e}_2$$

This vector represents the following function.

$$\begin{aligned}\left(\frac{2}{3}\pi^2\boldsymbol{e}_1 - 4\boldsymbol{e}_2\right)(x) &= \frac{2}{3}\pi^2 e_1(x) - 4e_2(x) \\ &= \frac{1}{3}\pi^2 - 4\cos x\end{aligned}$$

Notice that the function consists of the terms up to the terms of $\cos x$ and $\sin x$ in the Fourier series of $f(x)$.

Solution 2 Assume the following equation holds.

$$f = c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3$$

(There is actually no coefficients c_1, c_2, and c_3 that satisfy the equation, but it's ok.) Take the inner product with $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ in the both side.

$$\begin{aligned}(\boldsymbol{f}, \boldsymbol{e}_1) = c_1(\boldsymbol{e}_1, \boldsymbol{e}_1) + c_2(\boldsymbol{e}_2, \boldsymbol{e}_1) + c_3(\boldsymbol{e}_3, \boldsymbol{e}_1) \\ (\boldsymbol{f}, \boldsymbol{e}_2) = c_1(\boldsymbol{e}_1, \boldsymbol{e}_2) + c_2(\boldsymbol{e}_2, \boldsymbol{e}_2) + c_3(\boldsymbol{e}_3, \boldsymbol{e}_2) \\ (\boldsymbol{f}, \boldsymbol{e}_3) = c_1(\boldsymbol{e}_1, \boldsymbol{e}_3) + c_2(\boldsymbol{e}_2, \boldsymbol{e}_3) + c_3(\boldsymbol{e}_3, \boldsymbol{e}_3)\end{aligned}$$

We write the above three equations in the matrix form.

$$\begin{pmatrix}(\boldsymbol{e}_1, \boldsymbol{e}_1) & (\boldsymbol{e}_2, \boldsymbol{e}_1) & (\boldsymbol{e}_3, \boldsymbol{e}_1) \\ (\boldsymbol{e}_1, \boldsymbol{e}_2) & (\boldsymbol{e}_2, \boldsymbol{e}_2) & (\boldsymbol{e}_3, \boldsymbol{e}_2) \\ (\boldsymbol{e}_1, \boldsymbol{e}_3) & (\boldsymbol{e}_2, \boldsymbol{e}_3) & (\boldsymbol{e}_3, \boldsymbol{e}_3)\end{pmatrix}\begin{pmatrix}c_1 \\ c_2 \\ c_3\end{pmatrix} = \begin{pmatrix}(\boldsymbol{f}, \boldsymbol{e}_1) \\ (\boldsymbol{f}, \boldsymbol{e}_2) \\ (\boldsymbol{f}, \boldsymbol{e}_3)\end{pmatrix}$$

Notice that this is the same as the one obtained in Solution 1. So we omit the remaining calculation.

Solution 3

A linear combination of $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ is written as $c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3$. It is closest to f when it is the projection of f on the subspace spanned by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$. That is, the vector $f - (c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3)$ is orthogonal to the subspace spanned by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$. So the following three equations should hold.

$$\begin{aligned}(f - (c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3), \boldsymbol{e}_1) = 0 \\ (f - (c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3), \boldsymbol{e}_2) = 0 \\ (f - (c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3), \boldsymbol{e}_3) = 0\end{aligned}$$

We expand the inner product in each of the equations.

$$
\begin{aligned}
(\boldsymbol{f}, \boldsymbol{e}_1) - (c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3, \boldsymbol{e}_1) = 0 \\
(\boldsymbol{f}, \boldsymbol{e}_2) - (c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3, \boldsymbol{e}_2) = 0 \\
(\boldsymbol{f}, \boldsymbol{e}_3) - (c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3, \boldsymbol{e}_3) = 0
\end{aligned}
$$

We move the second inner product into RHS in each of the equations.

$$
\begin{aligned}
(\boldsymbol{f}, \boldsymbol{e}_1) = (c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3, \boldsymbol{e}_1) \\
(\boldsymbol{f}, \boldsymbol{e}_2) = (c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3, \boldsymbol{e}_2) \\
(\boldsymbol{f}, \boldsymbol{e}_3) = (c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3, \boldsymbol{e}_3)
\end{aligned}
$$

We expand the inner product in the RHS in each of the equations.

$$
\begin{aligned}
(\boldsymbol{f}, \boldsymbol{e}_1) = c_1(\boldsymbol{e}_1, \boldsymbol{e}_1) + c_2(\boldsymbol{e}_2, \boldsymbol{e}_1) + c_3(\boldsymbol{e}_3, \boldsymbol{e}_1) \\
(\boldsymbol{f}, \boldsymbol{e}_2) = c_1(\boldsymbol{e}_1, \boldsymbol{e}_2) + c_2(\boldsymbol{e}_2, \boldsymbol{e}_2) + c_3(\boldsymbol{e}_3, \boldsymbol{e}_2) \\
(\boldsymbol{f}, \boldsymbol{e}_3) = c_1(\boldsymbol{e}_1, \boldsymbol{e}_3) + c_2(\boldsymbol{e}_2, \boldsymbol{e}_3) + c_3(\boldsymbol{e}_3, \boldsymbol{e}_3)
\end{aligned}
$$

We write the above three equations in the matrix form.

$$
\begin{pmatrix}
(\boldsymbol{e}_1, \boldsymbol{e}_1) & (\boldsymbol{e}_2, \boldsymbol{e}_1) & (\boldsymbol{e}_3, \boldsymbol{e}_1) \\
(\boldsymbol{e}_1, \boldsymbol{e}_2) & (\boldsymbol{e}_2, \boldsymbol{e}_2) & (\boldsymbol{e}_3, \boldsymbol{e}_2) \\
(\boldsymbol{e}_1, \boldsymbol{e}_3) & (\boldsymbol{e}_2, \boldsymbol{e}_3) & (\boldsymbol{e}_3, \boldsymbol{e}_3)
\end{pmatrix}
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}
=
\begin{pmatrix} (\boldsymbol{f}, \boldsymbol{e}_1) \\ (\boldsymbol{f}, \boldsymbol{e}_2) \\ (\boldsymbol{f}, \boldsymbol{e}_3) \end{pmatrix}
$$

Notice that this is the same as the one obtained in Solution 1 and 2. So we omit the remaining calculation.