Supplement 13: the Euler formula

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The exponential function e^z (from complex numbers to complex numbers) is analytic for all z and $(e^z)' = e^z$. So we obtain the following Maclaurin series.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

By setting $z = iy$ in this equation we obtain the following equation.

$$\begin{aligned} e^{iy} &= \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(iy)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(iy)^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i(-1)^k y^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} y^{2k} + i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} y^{2k+1} \end{aligned}$$

The series on the right hand side are the Maclaurin series of $\cos y$ and $\sin y$.

$$\begin{aligned} \cos y &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} y^{2k} \\ \sin y &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} y^{2k+1} \end{aligned}$$

So we obtain the Euler formula.

$$e^{iy} = \cos y + i \sin y$$

(Note) Let us see the equality

$$\sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{k=0}^{\infty} \frac{(iy)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(iy)^{2k+1}}{(2k+1)!}$$

in the above argument. In general the value of the infinite series may not be the same if we change the order of summation. But the series

$$\sum_{n=0}^{\infty} \frac{(iy)^n}{n!}$$

absolutely converges and in such cases we can change freely the order of summation. So the equality holds.