Three solutions for Exercise 4

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Exercise Approximate a column vector $\boldsymbol{a} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$ by a linear combination of the column vectors $\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (i.e., $\sum_{k=1}^{2} c_k \boldsymbol{u}_k = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2$ for some c_1 and c_2). That is, obtain c_1 and c_2 so that $c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2$ is closest to $\boldsymbol{a}$. As for the measure of the distance, use (the half of) the square of the norm of the difference of $c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2$ and $\boldsymbol{a}$.

$$J = \frac{1}{2}\left\| \sum_{k=1}^{2} c_k \boldsymbol{u}_k - \boldsymbol{a} \right\|^2$$

The norm of a column vector $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is defined as follows.

$$\|\boldsymbol{x}\| = \sqrt{(\boldsymbol{x}, \boldsymbol{x})} = \sqrt{\sum_{k=1}^{3} x_k^2}$$

Solutions We show two solutions. One is by substituting the given column vectors into the normal equations and the other is by substituting them from the beginning. Solution 1 is clearer.

Solution 1 Firstly calculate J as follows.

$$J = \frac{1}{2}\left\| \sum_{k=1}^{2} c_k \boldsymbol{u}_k - \boldsymbol{a} \right\|^2$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\sum_{k=1}^{2} c_k \boldsymbol{u}_k - \boldsymbol{a}, \sum_{k=1}^{2} c_k \boldsymbol{u}_k - \boldsymbol{a}\right) \\
&= \frac{1}{2}\left\{\left(\sum_{k=1}^{2} c_k \boldsymbol{u}_k, \sum_{k=1}^{2} c_k \boldsymbol{u}_k\right) - 2\left(\boldsymbol{a}, \sum_{k=1}^{2} c_k \boldsymbol{u}_k\right) + \|\boldsymbol{a}\|^2\right\} \\
&= \frac{1}{2}\left\{\sum_{k,l=1}^{2} c_k c_l (\boldsymbol{u}_k, \boldsymbol{u}_l) - 2\sum_{k=1}^{2} c_k (\boldsymbol{a}, \boldsymbol{u}_k) + \|\boldsymbol{a}\|^2\right\}
\end{aligned}
$$

Partially differentiate this with respect to c_i $(i = 1, 2)$.

$$
\begin{aligned}
\frac{\partial J}{\partial c_i} &= \frac{\partial}{\partial c_i}\frac{1}{2}\left\{\sum_{k,l=1}^{2} c_k c_l (\boldsymbol{u}_k, \boldsymbol{u}_l) - 2\sum_{k=1}^{2} c_k (\boldsymbol{a}, \boldsymbol{u}_k) + \|\boldsymbol{a}\|^2\right\} \\
&= \frac{1}{2}\left\{\frac{\partial}{\partial c_i}\sum_{k,l=1}^{2} c_k c_l (\boldsymbol{u}_k, \boldsymbol{u}_l) - 2\frac{\partial}{\partial c_i}\sum_{k=1}^{2} c_k (\boldsymbol{a}, \boldsymbol{u}_k)\right\} \\
&= \frac{1}{2}\left\{2\sum_{k=1}^{2} c_k (\boldsymbol{u}_k, \boldsymbol{u}_i) - 2(\boldsymbol{a}, \boldsymbol{u}_i)\right\} \\
&= \sum_{k=1}^{2} c_k (\boldsymbol{u}_k, \boldsymbol{u}_i) - (\boldsymbol{a}, \boldsymbol{u}_i)
\end{aligned}
$$

By writing $\dfrac{\partial J}{\partial c_1} = 0$ and $\dfrac{\partial J}{\partial c_2} = 0$ in matrix form, we obtain

$$
\begin{pmatrix} (\boldsymbol{u}_1, \boldsymbol{u}_1) & (\boldsymbol{u}_2, \boldsymbol{u}_1) \\ (\boldsymbol{u}_1, \boldsymbol{u}_2) & (\boldsymbol{u}_2, \boldsymbol{u}_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} (\boldsymbol{a}, \boldsymbol{u}_1) \\ (\boldsymbol{a}, \boldsymbol{u}_2) \end{pmatrix}
$$

By substituting column vectors $\boldsymbol{a}$, $\boldsymbol{u}_1$, and $\boldsymbol{u}_2$ in the above equation we obtain

$$
\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \end{pmatrix}
$$

By solving this we obtain

$$
\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}
$$

Thus the linear combination of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ that is closest to the vector $\boldsymbol{a}$ is obtained as follows.

$$
4\boldsymbol{u}_1 - \boldsymbol{u}_2 = 4\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}
$$

Solution 2 By substituting $\boldsymbol{a}$, $\boldsymbol{u}_1$, and $\boldsymbol{u}_2$ in J we obtain

$$\begin{aligned}
J &= \frac{1}{2}\|c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 - \boldsymbol{a}\|^2 \\
&= \frac{1}{2}\left\|c_1\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}3\\2\\6\end{pmatrix}\right\|^2 \\
&= \frac{1}{2}\left\|\begin{pmatrix}c_1+c_2-3\\c_1-2\\c_1-6\end{pmatrix}\right\|^2 \\
&= \frac{1}{2}\left\{c_1^2+c_2^2+9+2c_1c_2-6c_1-6c_2+c_1^2-4c_1+4+c_1^2-12c_1+36\right\} \\
&= \frac{1}{2}\left\{3c_1^2+c_2^2+2c_1c_2-22c_1-6c_2+49\right\}
\end{aligned}$$

Partially differentiate this with respect to c_1 and c_2.

$$\begin{aligned}
\frac{\partial J}{\partial c_1} &= \frac{1}{2}\{6c_1+2c_2-22\} = 3c_1+c_2-11 \\
\frac{\partial J}{\partial c_2} &= \frac{1}{2}\{2c_1+2c_2-6\} = c_1+c_2-3
\end{aligned}$$

Then we obtain the following systems of equations.

$$\begin{aligned}
3c_1+c_2 &= 11 \\
c_1+c_2 &= 3
\end{aligned}$$

By solving this we obtain $c_1 = 4, c_2 = -1$. Thus the linear combination of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ that is closest to the vector $\boldsymbol{a}$ is obtained as follows.

$$4\boldsymbol{u}_1 - \boldsymbol{u}_2 = 4\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}3\\4\\4\end{pmatrix}$$

Solution 3 The set of linear combinations of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2$$

constitutes the subspace spanned by $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. This subspace is a plane in the three dimensional space and the norm of the difference between the vector $\boldsymbol{a}$ and a linear combination $c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2$ is smallest when the difference and $\boldsymbol{u}_1$ are orthogonal and the difference and $\boldsymbol{u}_2$ are orthogonal.

$$(c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 - \boldsymbol{a}, \boldsymbol{u}_1) = 0$$
$$(c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 - \boldsymbol{a}, \boldsymbol{u}_2) = 0$$

The first equation is calculated as follows.

$$\begin{aligned}
(c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 - \boldsymbol{a}, \boldsymbol{u}_1) &= (c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}) \\
&= (\begin{pmatrix} c_1 + c_2 - 3 \\ c_1 - 2 \\ c_1 - 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}) \\
&= c_1 + c_2 - 3 + c_1 - 2 + c_1 - 6 \\
&= 3c_1 + c_2 - 11 \\
&= 0
\end{aligned}$$

The second equation is calculated as follows.

$$\begin{aligned}
(c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 - \boldsymbol{a}, \boldsymbol{u}_2) &= (c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) \\
&= (\begin{pmatrix} c_1 + c_2 - 3 \\ c_1 - 2 \\ c_1 - 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) \\
&= c_1 + c_2 - 3 \\
&= 0
\end{aligned}$$

By solving these equations we obtain $c_1 = 4, c_2 = -1$. Thus the linear combination of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ that is closest to the vector $\boldsymbol{a}$ is obtained as follows.

$$4\boldsymbol{u}_1 - \boldsymbol{u}_2 = 4 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$$