

# Solutions for Mid-term examination

Isao Sasano

May 30, 2017

**Problem 1 (10 points)** Fit a straight line (a linear function) to the three points  $(0, 0)$ ,  $(1, 1)$ ,  $(3, 4)$  so that the half of the sum of the squares of the distances of those points from the straight line is minimum, where the distance is measured in the vertical direction (the y-direction).

**Solution** Let the line (the linear function) be  $f(x) = ax + b$  and  $(x_1, y_1) = (0, 0)$ ,  $(x_2, y_2) = (1, 1)$ ,  $(x_3, y_3) = (3, 4)$ . The half of the sum of the squares of the distances of these points from the line is given as follows.

$$J = \frac{1}{2} \sum_{i=1}^3 (f(x_i) - y_i)^2 = \frac{1}{2} \sum_{i=1}^3 (ax_i + b - y_i)^2$$

$J$  takes the minimum value in the point where the partial derivatives of  $J$  with respect to  $a$  and  $b$  are 0.

$$\frac{\partial J}{\partial a} = 0, \quad \frac{\partial J}{\partial b} = 0$$

Firstly the partial derivative of  $J$  with respect to  $a$  is calculated as follows.

$$\begin{aligned} \frac{\partial J}{\partial a} &= \frac{\partial}{\partial a} \left\{ \frac{1}{2} \sum_{i=1}^3 (ax_i + b - y_i)^2 \right\} \\ &= \frac{1}{2} \sum_{i=1}^3 \frac{\partial}{\partial a} (ax_i + b - y_i)^2 \\ &= \frac{1}{2} \sum_{i=1}^3 2(ax_i + b - y_i)x_i \\ &= \sum_{i=1}^3 (ax_i + b - y_i)x_i \\ &= \sum_{i=1}^3 (ax_i^2 + bx_i - x_i y_i) \\ &= a \sum_{i=1}^3 x_i^2 + b \sum_{i=1}^3 x_i - \sum_{i=1}^3 x_i y_i \end{aligned}$$

Secondly the partial derivative of  $J$  with respect to  $b$  is calculated as follows.

$$\frac{\partial J}{\partial b} = \frac{\partial}{\partial b} \left\{ \frac{1}{2} \sum_{i=1}^3 (ax_i + b - y_i)^2 \right\}$$

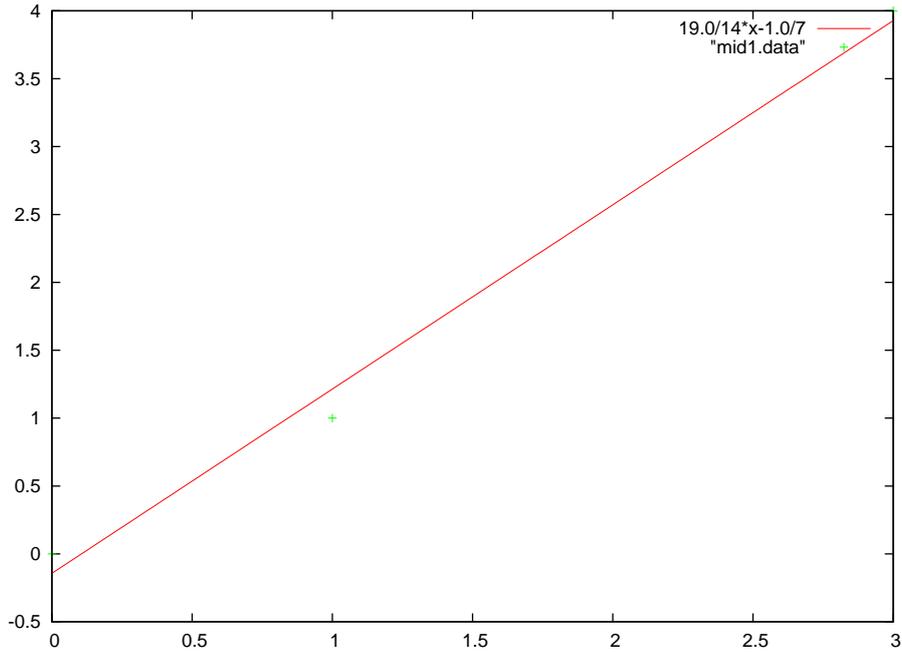


Figure 1: The straight line closest to the given three points

$$\begin{aligned}
 &= \frac{1}{2} \sum_{i=1}^3 \frac{\partial}{\partial b} (ax_i + b - y_i)^2 \\
 &= \frac{1}{2} \sum_{i=1}^3 2(ax_i + b - y_i) \\
 &= \sum_{i=1}^3 (ax_i + b - y_i) \\
 &= a \sum_{i=1}^3 x_i + b \sum_{i=1}^3 1 - \sum_{i=1}^3 y_i
 \end{aligned}$$

Then we obtain the system of equations

$$\begin{aligned}
 10a + 4b - 13 &= 0 \\
 4a + 3b - 5 &= 0
 \end{aligned}$$

and  $a = \frac{19}{14}, b = -\frac{1}{7}$  is the solution. Hence the function is obtained as follows.

$$f(x) = \frac{19}{14}x - \frac{1}{7}$$

**Supplement:** The function is depicted with the three points in Fig. 1.

**Problem 2 (10 points)** Fit a parabola (a square function) to the four points  $(-1, 0), (0, -1), (1, 0), (2, 1)$  so that the half of the sum of the squares of the distances of those points from the parabola is minimum, where the distance is measured in the vertical direction (the y-direction).

**Solution** Let the function be  $f(x) = ax^2 + bx + c$  and  $(x_1, y_1) = (-1, 0), (x_2, y_2) = (0, -1), (x_3, y_3) = (1, 0), (x_4, y_4) = (2, 1)$ . The half of the

sum of the squares of the distances of these points from the parabola is given as follows.

$$J = \frac{1}{2} \sum_{i=1}^4 (f(x_i) - y_i)^2 = \frac{1}{2} \sum_{i=1}^4 (ax_i^2 + bx_i + c - y_i)^2$$

$J$  takes the minimum value in the point where the partial derivatives of  $J$  with respect to  $a$ ,  $b$ , and  $c$  are 0.

$$\frac{\partial J}{\partial a} = 0, \quad \frac{\partial J}{\partial b} = 0, \quad \frac{\partial J}{\partial c} = 0$$

Firstly the partial derivative of  $J$  with respect to  $a$  is calculated as follows.

$$\begin{aligned} \frac{\partial J}{\partial a} &= \frac{\partial}{\partial a} \left\{ \frac{1}{2} \sum_{i=1}^4 (ax_i^2 + bx_i + c - y_i)^2 \right\} \\ &= \frac{1}{2} \sum_{i=1}^4 \frac{\partial}{\partial a} (ax_i^2 + bx_i + c - y_i)^2 \\ &= \frac{1}{2} \sum_{i=1}^4 2(ax_i^2 + bx_i + c - y_i) \frac{\partial}{\partial a} (ax_i^2 + bx_i + c - y_i) \\ &= \frac{1}{2} \sum_{i=1}^4 2(ax_i^2 + bx_i + c - y_i)x_i^2 \\ &= \sum_{i=1}^4 (ax_i^2 + bx_i + c - y_i)x_i^2 \\ &= \sum_{i=1}^4 (ax_i^4 + bx_i^3 + cx_i^2 - x_i^2y_i) \\ &= a \sum_{i=1}^4 x_i^4 + b \sum_{i=1}^4 x_i^3 + c \sum_{i=1}^4 x_i^2 - \sum_{i=1}^4 x_i^2y_i \end{aligned}$$

Secondly the partial derivative of  $J$  with respect to  $b$  is calculated as follows.

$$\begin{aligned} \frac{\partial J}{\partial b} &= \frac{\partial}{\partial b} \left\{ \frac{1}{2} \sum_{i=1}^4 (ax_i^2 + bx_i + c - y_i)^2 \right\} \\ &= \frac{1}{2} \sum_{i=1}^4 \frac{\partial}{\partial b} (ax_i^2 + bx_i + c - y_i)^2 \\ &= \frac{1}{2} \sum_{i=1}^4 2(ax_i^2 + bx_i + c - y_i) \frac{\partial}{\partial b} (ax_i^2 + bx_i + c - y_i) \\ &= \frac{1}{2} \sum_{i=1}^4 2(ax_i^2 + bx_i + c - y_i)x_i \\ &= \sum_{i=1}^4 (ax_i^2 + bx_i + c - y_i)x_i \\ &= \sum_{i=1}^4 (ax_i^3 + bx_i^2 + cx_i - x_iy_i) \\ &= a \sum_{i=1}^4 x_i^3 + b \sum_{i=1}^4 x_i^2 + c \sum_{i=1}^4 x_i - \sum_{i=1}^4 x_iy_i \end{aligned}$$

Thirdly the partial derivative of  $J$  with respect to  $c$  is calculated as follows.

$$\begin{aligned}
\frac{\partial J}{\partial c} &= \frac{\partial}{\partial c} \left\{ \frac{1}{2} \sum_{i=1}^4 (ax_i^2 + bx_i + c - y_i)^2 \right\} \\
&= \frac{1}{2} \sum_{i=1}^4 \frac{\partial}{\partial c} (ax_i^2 + bx_i + c - y_i)^2 \\
&= \frac{1}{2} \sum_{i=1}^4 2(ax_i^2 + bx_i + c - y_i) \frac{\partial}{\partial c} (ax_i^2 + bx_i + c - y_i) \\
&= \frac{1}{2} \sum_{i=1}^4 2(ax_i^2 + bx_i + c - y_i) \cdot 1 \\
&= \sum_{i=1}^4 (ax_i^2 + bx_i + c - y_i) \\
&= a \sum_{i=1}^4 x_i^2 + b \sum_{i=1}^4 x_i + c \sum_{i=1}^4 1 - \sum_{i=1}^4 y_i
\end{aligned}$$

Then we obtain the system of equations with respect to  $a$ ,  $b$ , and  $c$ . The coefficients of the equations are computed as follows.

$$\begin{aligned}
\sum_{i=1}^4 x_i^4 = 18, \quad \sum_{i=1}^4 x_i^3 = 8, \quad \sum_{i=1}^4 x_i^2 = 6, \quad \sum_{i=1}^4 x_i = 2 \\
\sum_{i=1}^4 1 = 4, \quad \sum_{i=1}^4 x_i^2 y_i = 4, \quad \sum_{i=1}^4 x_i y_i = 2, \quad \sum_{i=1}^4 y_i = 0
\end{aligned}$$

Hence the system of equations is obtained as follows.

$$\begin{aligned}
18a + 8b + 6c - 4 &= 0 \quad \dots (1) \\
8a + 6b + 2c - 2 &= 0 \quad \dots (2) \\
6a + 2b + 4c &= 0 \quad \dots (3)
\end{aligned}$$

By solving this, we obtain the solution.

$$a = \frac{1}{2}, \quad b = -\frac{1}{10}, \quad c = -\frac{7}{10}$$

Hence the function is obtained as follows.

$$f(x) = \frac{1}{2}x^2 - \frac{1}{10}x - \frac{7}{10}$$

**Supplement:** The function is depicted with the four points in Fig. 2. In Fig. 2 the green symbols are the given points and the red curve is the square function.

**Problem 3 (10 points)** Approximate a column vector  $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$  by a linear combination of the column vectors  $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$  and  $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

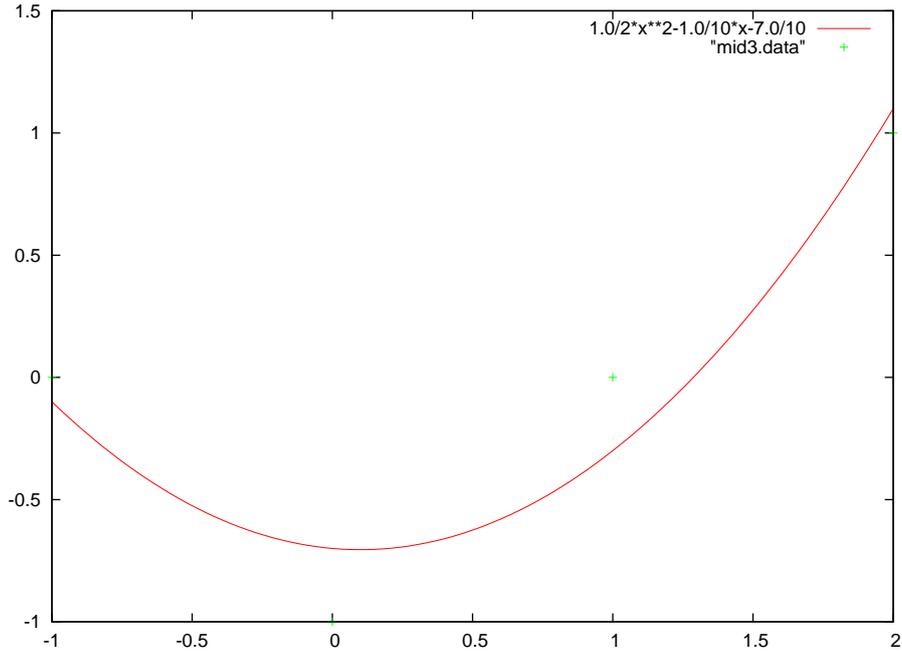


Figure 2: The parabola closest to the given four points

(i.e.,  $\sum_{k=1}^2 c_k \mathbf{u}_k = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2$  for some  $c_1$  and  $c_2$ ). As for the measure of the distance, use the half of the square of the norm of the difference of  $c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2$  and  $\mathbf{a}$ :  $J = \frac{1}{2} \left\| \sum_{k=1}^2 c_k \mathbf{u}_k - \mathbf{a} \right\|^2$ . The norm of a column vector  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$  is

defined to be  $\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})} = \sqrt{\sum_{k=1}^3 x_k^2}$ . You may use the normal equation.

$$\begin{pmatrix} (\mathbf{u}_1, \mathbf{u}_1) & (\mathbf{u}_2, \mathbf{u}_1) \\ (\mathbf{u}_1, \mathbf{u}_2) & (\mathbf{u}_2, \mathbf{u}_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} (\mathbf{a}, \mathbf{u}_1) \\ (\mathbf{a}, \mathbf{u}_2) \end{pmatrix}$$

**Solutions** We show two solutions. One is by substituting the given column vectors into the normal equation and the other is by substituting them from the beginning. Solution 1 is clearer.

**Solution 1** Firstly calculate  $J$  as follows.

$$\begin{aligned} J &= \frac{1}{2} \left\| \sum_{k=1}^2 c_k \mathbf{u}_k - \mathbf{a} \right\|^2 \\ &= \frac{1}{2} \left( \sum_{k=1}^2 c_k \mathbf{u}_k - \mathbf{a}, \sum_{k=1}^2 c_k \mathbf{u}_k - \mathbf{a} \right) \\ &= \frac{1}{2} \left\{ \left( \sum_{k=1}^2 c_k \mathbf{u}_k, \sum_{k=1}^2 c_k \mathbf{u}_k \right) - 2 \left( \mathbf{a}, \sum_{k=1}^2 c_k \mathbf{u}_k \right) + \|\mathbf{a}\|^2 \right\} \\ &= \frac{1}{2} \left\{ \sum_{k,l=1}^2 c_k c_l (\mathbf{u}_k, \mathbf{u}_l) - 2 \sum_{k=1}^2 c_k (\mathbf{a}, \mathbf{u}_k) + \|\mathbf{a}\|^2 \right\} \end{aligned}$$

Partially differentiate this with respect to  $c_i$  ( $i = 1, 2$ ).

$$\begin{aligned}
\frac{\partial J}{\partial c_i} &= \frac{\partial}{\partial c_i} \frac{1}{2} \left\{ \sum_{k,l=1}^2 c_k c_l (\mathbf{u}_k, \mathbf{u}_l) - 2 \sum_{k=1}^2 c_k (\mathbf{a}, \mathbf{u}_k) + \|\mathbf{a}\|^2 \right\} \\
&= \frac{1}{2} \left\{ \frac{\partial}{\partial c_i} \sum_{k,l=1}^2 c_k c_l (\mathbf{u}_k, \mathbf{u}_l) - 2 \frac{\partial}{\partial c_i} \sum_{k=1}^2 c_k (\mathbf{a}, \mathbf{u}_k) \right\} \\
&= \frac{1}{2} \left\{ 2 \sum_{k=1}^2 c_k (\mathbf{u}_k, \mathbf{u}_i) - 2 (\mathbf{a}, \mathbf{u}_i) \right\} \\
&= \sum_{k=1}^2 c_k (\mathbf{u}_k, \mathbf{u}_i) - (\mathbf{a}, \mathbf{u}_i)
\end{aligned}$$

By writing  $\frac{\partial J}{\partial c_1} = 0$  and  $\frac{\partial J}{\partial c_2} = 0$  in matrix form, we obtain the normal equation.

$$\begin{pmatrix} (\mathbf{u}_1, \mathbf{u}_1) & (\mathbf{u}_2, \mathbf{u}_1) \\ (\mathbf{u}_1, \mathbf{u}_2) & (\mathbf{u}_2, \mathbf{u}_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} (\mathbf{a}, \mathbf{u}_1) \\ (\mathbf{a}, \mathbf{u}_2) \end{pmatrix}$$

By substituting column vectors  $\mathbf{a}$ ,  $\mathbf{u}_1$ , and  $\mathbf{u}_2$  in the above equation we obtain

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \end{pmatrix}$$

By solving this we obtain

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

Thus the linear combination of  $\mathbf{u}_1$  and  $\mathbf{u}_2$  that is closest to the vector  $\mathbf{a}$  is obtained as follows.

$$4\mathbf{u}_1 - \mathbf{u}_2 = 4 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$$

**Solution 2** By substituting  $\mathbf{a}$ ,  $\mathbf{u}_1$ , and  $\mathbf{u}_2$  in  $J$  we obtain

$$\begin{aligned}
J &= \frac{1}{2} \|c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 - \mathbf{a}\|^2 \\
&= \frac{1}{2} \left\| c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \right\|^2 \\
&= \frac{1}{2} \left\| \begin{pmatrix} c_1 + c_2 - 3 \\ c_1 - 2 \\ c_1 - 6 \end{pmatrix} \right\|^2 \\
&= \frac{1}{2} \{ c_1^2 + c_2^2 + 9 + 2c_1 c_2 - 6c_1 - 6c_2 + c_1^2 - 4c_1 + 4 + c_1^2 - 12c_1 + 36 \} \\
&= \frac{1}{2} \{ 3c_1^2 + c_2^2 + 2c_1 c_2 - 22c_1 - 6c_2 + 49 \}
\end{aligned}$$

Partially differentiate this with respect to  $c_1$  and  $c_2$ .

$$\begin{aligned}\frac{\partial J}{\partial c_1} &= \frac{1}{2}\{6c_1 + 2c_2 - 22\} = 3c_1 + c_2 - 11 \\ \frac{\partial J}{\partial c_2} &= \frac{1}{2}\{2c_1 + 2c_2 - 6\} = c_1 + c_2 - 3\end{aligned}$$

Then we obtain the following systems of equations.

$$\begin{aligned}3c_1 + c_2 &= 11 \\ c_1 + c_2 &= 3\end{aligned}$$

By solving this we obtain  $c_1 = 4, c_2 = -1$ . Thus the linear combination of  $\mathbf{u}_1$  and  $\mathbf{u}_2$  that is closest to the vector  $\mathbf{a}$  is obtained as follows.

$$4\mathbf{u}_1 - \mathbf{u}_2 = 4 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$$

**Problem 4 (10 points)** Calculate the following integral.

$$\int_{-\pi}^{\pi} \cos^2 x dx$$

### Solution

By the additive formula of  $\cos$  we obtain

$$\begin{aligned}\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ \cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta\end{aligned}$$

By adding these equations we obtain

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta$$

By dividing this equation by two we obtain

$$\cos \alpha \cos \beta = \frac{1}{2}\{\cos(\alpha + \beta) + \cos(\alpha - \beta)\}$$

By letting  $\alpha$  and  $\beta$  be  $x$  we obtain

$$\begin{aligned}\cos^2 x &= \frac{1}{2}(\cos 2x + \cos 0) \\ &= \frac{1}{2}(\cos 2x + 1)\end{aligned}$$

By integrate this equation we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} \cos^2 x dx &= \int_{-\pi}^{\pi} \frac{1}{2}(\cos 2x + 1) dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \cos 2x + 1 dx \\ &= \frac{1}{2} \left\{ \int_{-\pi}^{\pi} \cos 2x dx + \int_{-\pi}^{\pi} 1 dx \right\} \\ &= \frac{1}{2}(0 + 2\pi) \\ &= \pi\end{aligned}$$