

Supplement 9:

A derivation of a step in the proof of Triangle inequality

Isao Sasano

June 14, 2016

This material shows the following step in the proof of Triangle inequality by using the axioms for inner product.

$$(\mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}) = (\mathbf{u}, \mathbf{u}) + 2(\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{v})$$

The axioms for inner product are as follows. Let the inner product space be $\mathcal{L}$.

1. Positive definiteness:
 $\forall \mathbf{u} \in \mathcal{L}. (\mathbf{u}, \mathbf{u}) \geq 0$. The equality $(\mathbf{u}, \mathbf{u}) = 0$ holds iff $\mathbf{u} = \mathbf{0}$.
2. Symmetry:
 $\forall \mathbf{u}, \mathbf{v} \in \mathcal{L}. (\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$.
3. Linearity:
 $\forall c_1, c_2 \in \mathbb{R} \wedge \forall \mathbf{u}_1, \mathbf{u}_2, \mathbf{v} \in \mathcal{L}. (c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2, \mathbf{v}) = c_1(\mathbf{u}_1, \mathbf{v}) + c_2(\mathbf{u}_2, \mathbf{v})$.

We derive RHS from LHS as follows.

$$\begin{aligned} & (\mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}) \\ = & \quad \{ \text{linearity} \} \\ & (\mathbf{u}, \mathbf{u} + \mathbf{v}) + (\mathbf{v}, \mathbf{u} + \mathbf{v}) \\ = & \quad \{ \text{symmetry} \} \\ & (\mathbf{u} + \mathbf{v}, \mathbf{u}) + (\mathbf{u} + \mathbf{v}, \mathbf{v}) \\ = & \quad \{ \text{linearity} \} \\ & (\mathbf{u}, \mathbf{u}) + (\mathbf{v}, \mathbf{u}) + (\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{v}) \\ = & \quad \{ \text{symmetry} \} \\ & (\mathbf{u}, \mathbf{u}) + (\mathbf{u}, \mathbf{v}) + (\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{v}) \\ = & \quad \{ \text{arithmetic of real numbers} \} \\ & (\mathbf{u}, \mathbf{u}) + 2(\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{v}) \end{aligned}$$