Example 3

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Example Fit a line (a linear function) to the function $\sin x$ on $[0, \frac{\pi}{2}]$ so that (the half of) the integral of the squares of the distances between them, where the distance is measured in the vertical direction (the y-direction).
Solution Let the function be $f(x) = ax + b$. The half of the integral of the squares of the distances between $f(x)$ and $\sin x$ on $[0, \frac{\pi}{2}]$ is given as follows.

$$\begin{aligned} J &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \{f(x) - \sin x\}^2 \mathrm{d}x \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \{ax + b - \sin x\}^2 \mathrm{d}x \end{aligned}$$

J takes the minimum value in the point where the partial derivatives of J with respect to a, b, and c are 0.

$$\frac{\partial J}{\partial a} = 0, \quad \frac{\partial J}{\partial b} = 0$$

Firstly the partial derivative of J with respect to a is calculated as follows.

$$\begin{aligned} \frac{\partial J}{\partial a} &= \frac{\partial}{\partial a}\frac{1}{2}\int_0^{\frac{\pi}{2}} \{ax + b - \sin x\}^2 \mathrm{d}x \\ &= \frac{1}{2}\frac{\partial}{\partial a}\int_0^{\frac{\pi}{2}} \{ax + b - \sin x\}^2 \mathrm{d}x \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\partial}{\partial a}\{ax + b - \sin x\}^2 \mathrm{d}x \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\{ax + b - \sin x\} x \mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \{ax^2 + bx - x\sin x\} \mathrm{d}x \\ &= a\int_0^{\frac{\pi}{2}} x^2 \mathrm{d}x + b\int_0^{\frac{\pi}{2}} x \mathrm{d}x - \int_0^{\frac{\pi}{2}} x\sin x \mathrm{d}x \end{aligned}$$

Here we calculate each of the integrals. As for x^2 we obtain

$$\int_0^{\frac{\pi}{2}} x^2 \mathrm{d}x = \left[\frac{x^3}{3}\right]_0^{\frac{\pi}{2}} = \frac{\pi^3}{24}$$

and as for x we obtain

$$\int_0^{\frac{\pi}{2}} x \mathrm{d}x = \left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{8}.$$

In the following we calculate $\int_0^{\frac{\pi}{2}} x \sin x \mathrm{d}x$.

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x \sin x \mathrm{d}x &= \left[x \frac{\cos x}{-1}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\cos x}{-1} \mathrm{d}x \\ &= [\sin x]_0^{\frac{\pi}{2}} \\ &= 1 \end{aligned}$$

Thus $\frac{\partial J}{\partial a}$ is obtained as follows.

$$\frac{\partial J}{\partial a} = \frac{\pi^3}{24} a + \frac{\pi^2}{8} b - 1$$

Secondly the partial derivative of J with respect to b is calculated as follows.

$$\begin{aligned} \frac{\partial J}{\partial b} &= \frac{\partial}{\partial b} \frac{1}{2} \int_0^{\frac{\pi}{2}} \{ax + b - \sin x\}^2 \mathrm{d}x \\ &= \frac{1}{2} \frac{\partial}{\partial b} \int_0^{\frac{\pi}{2}} \{ax + b - \sin x\}^2 \mathrm{d}x \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial b} \{ax + b - \sin x\}^2 \mathrm{d}x \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} 2\{ax + b - \sin x\} \mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \{ax + b - \sin x\} \mathrm{d}x \\ &= a \int_0^{\frac{\pi}{2}} x \mathrm{d}x + b \int_0^{\frac{\pi}{2}} \mathrm{d}x - \int_0^{\frac{\pi}{2}} \sin x \mathrm{d}x \\ &= \frac{\pi^2}{8} a + \frac{\pi}{2} b - 1 \end{aligned}$$

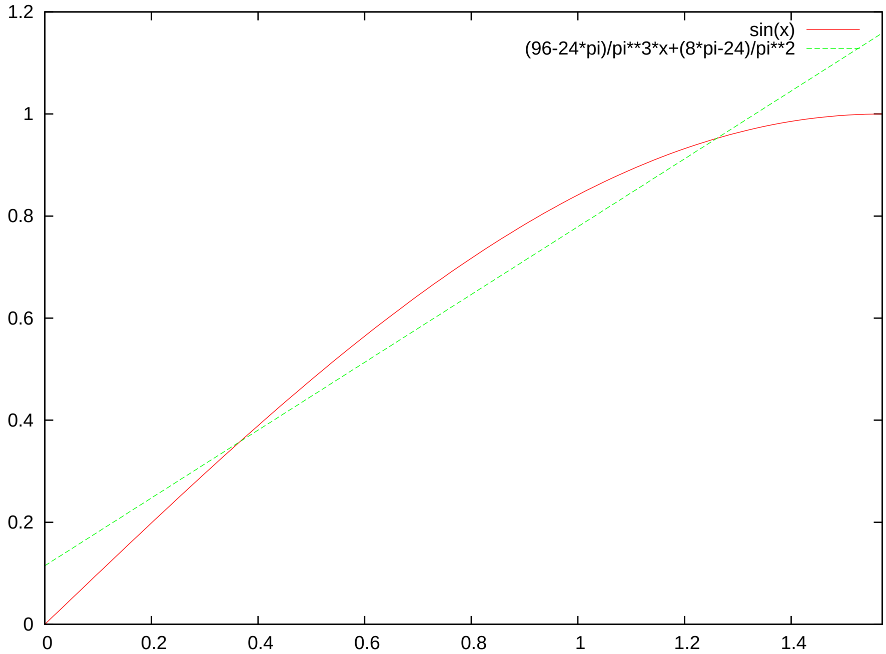

Figure 1: The closest linear function to $\sin x$ on the range $[0, \frac{\pi}{2}]$

Thus we obtain the system of equations with respect to a and b.

$$\frac{\pi^3}{24}a + \frac{\pi^2}{8}b - 1 = 0$$
$$\frac{\pi^2}{8}a + \frac{\pi}{2}b - 1 = 0$$

By solving this, we obtain the solution.

$$a = \frac{96 - 24\pi}{\pi^3}, \quad b = \frac{8\pi - 24}{\pi^2}$$

Hence the function is obtained as follows.

$$f(x) = \frac{96 - 24\pi}{\pi^3}x + \frac{8\pi - 24}{\pi^2}$$

The function is depicted with $\sin x$ on $[0, \frac{\pi}{2}]$ in Fig. 1. In Fig. 1 the red curve is the linear function and the green curve is the function $\sin x$.