

Exercise 9-2

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Exercise On the inner product space in Exercise 9-1, for the functions $\mathbf{f}_1(x) = x$ and $\mathbf{f}_2(x) = x^2$ calculate the values of $\|\mathbf{f}_1 + \mathbf{f}_2\|$ and $\|\mathbf{f}_1\| + \|\mathbf{f}_2\|$ and compare them.

Solution

$$\begin{aligned}\|\mathbf{f}_1 + \mathbf{f}_2\| &= \sqrt{(\mathbf{f}_1 + \mathbf{f}_2, \mathbf{f}_1 + \mathbf{f}_2)} \\&= \sqrt{\int_0^1 \{(\mathbf{f}_1 + \mathbf{f}_2)(x)\} \{(\mathbf{f}_1 + \mathbf{f}_2)(x)\} dx} \\&= \sqrt{\int_0^1 \{f_1(x) + f_2(x)\} \{f_1(x) + f_2(x)\} dx} \\&= \sqrt{\int_0^1 (x + x^2)(x + x^2) dx} \\&= \sqrt{\int_0^1 x^2 + 2x^3 + x^4 dx} \\&= \sqrt{\left[\frac{x^3}{3} + \frac{2x^4}{4} + \frac{x^5}{5} \right]_0^1} \\&= \sqrt{\frac{1}{3} + \frac{1}{2} + \frac{1}{5}} \\&= \sqrt{\frac{10 + 15 + 6}{30}} \\&= \sqrt{\frac{31}{30}} \\ \|\mathbf{f}_1\| + \|\mathbf{f}_2\| &= \sqrt{(\mathbf{f}_1, \mathbf{f}_1)} + \sqrt{(\mathbf{f}_2, \mathbf{f}_2)}\end{aligned}$$

$$\begin{aligned}
&= \sqrt{\int_0^1 x^2 dx} + \sqrt{\int_0^1 x^4 dx} \\
&= \sqrt{\left[\frac{x^3}{3}\right]_0^1} + \sqrt{\left[\frac{x^5}{5}\right]_0^1} \\
&= \sqrt{\frac{1}{3}} + \sqrt{\frac{1}{5}}
\end{aligned}$$

We compare $\|\mathbf{f}_1 + \mathbf{f}_2\|^2$ and $(\|\mathbf{f}_1\| + \|\mathbf{f}_2\|)^2$.

$$\begin{aligned}
(\|\mathbf{f}_1\| + \|\mathbf{f}_2\|)^2 - \|\mathbf{f}_1 + \mathbf{f}_2\|^2 &= \left(\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{5}}\right)^2 - \left(\sqrt{\frac{31}{30}}\right)^2 \\
&= \frac{1}{3} + \frac{1}{5} + 2\sqrt{\frac{1}{15}} - \frac{31}{30} \\
&= \frac{10 + 6 - 31}{30} + 2\sqrt{\frac{1}{15}} \\
&= -\frac{15}{30} + 2\sqrt{\frac{1}{15}} \\
&= 2\sqrt{\frac{1}{15}} - \frac{1}{2}
\end{aligned}$$

Here we compare $\left(2\sqrt{\frac{1}{15}}\right)^2$ and $\left(\frac{1}{2}\right)^2$.

$$\begin{aligned}
\left(2\sqrt{\frac{1}{15}}\right)^2 - \left(\frac{1}{2}\right)^2 &= \frac{4}{15} - \frac{1}{4} \\
&= \frac{16 - 15}{60} \\
&= \frac{1}{60} \\
&\geq 0
\end{aligned}$$

So we obtain $2\sqrt{\frac{1}{15}} - \frac{1}{2} \geq 0$ and hence the following inequality.

$$\|\mathbf{f}_1\| + \|\mathbf{f}_2\| > \|\mathbf{f}_1 + \mathbf{f}_2\|$$

We have shown that the Triangle inequality holds for the above example.