

A solution for Exercise 6

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Exercise Calculate the Fourier series expansion of the function $f(x) = x^2$ on the range $[-\pi, \pi]$.

Solution Assume the following equation holds. (Note: There are no coefficients $a_0, \dots, a_n, b_1, \dots, b_n$ that satisfy the equation, but it's ok.)

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx) \quad (1)$$

Integrate the both sides of the equation (1) on the range $[-\pi, \pi]$.

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) dx &= \int_{-\pi}^{\pi} \frac{1}{2} a_0 dx \\ &= a_0 \pi \end{aligned}$$

Then we calculate a_0 as follows.

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx \\ &= \frac{2}{3} \pi^2 \end{aligned}$$

Multiply the both sides of the equation (1) by $\cos kx$ and integrate them on the range $[-\pi, \pi]$.

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos kx dx &= a_k \int_{-\pi}^{\pi} \cos^2 kx dx \\ &= a_k \pi \end{aligned}$$

Then we calculate a_k as follows.

$$\begin{aligned}
a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos kx dx \\
&= \frac{1}{\pi} \left\{ \left[x^2 \frac{\sin kx}{k} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \frac{\sin kx}{k} dx \right\} \\
&= -\frac{2}{\pi k} \int_{-\pi}^{\pi} x \sin kx dx \quad \left(\text{since } \left[x^2 \frac{\sin kx}{k} \right]_{-\pi}^{\pi} \text{ is } 0 \right)
\end{aligned}$$

Here we calculate the integral $\int_{-\pi}^{\pi} x \sin kx dx$.

$$\begin{aligned}
\int_{-\pi}^{\pi} x \sin kx dx &= \left[x \frac{-\cos kx}{k} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{-\cos kx}{k} dx \\
&= -\frac{1}{k} [x \cos kx]_{-\pi}^{\pi} \quad \left(\text{since } \int_{-\pi}^{\pi} \frac{-\cos kx}{k} dx \text{ is } 0 \right) \\
&= -\frac{1}{k} (\pi \cos \pi k - (-\pi) \cos(-\pi k)) \\
&= -\frac{1}{k} (\pi \cos \pi k + \pi \cos \pi k) \\
&= -\frac{2\pi}{k} \cos \pi k \\
&= -\frac{2\pi}{k} (-1)^k
\end{aligned}$$

We resume the calculation of a_k .

$$\begin{aligned}
a_k &= -\frac{2}{\pi k} \left(-\frac{2\pi}{k} (-1)^k \right) \\
&= \frac{4}{k^2} (-1)^k
\end{aligned}$$

Multiply the both sides of the equation (1) by $\sin kx$ and integrate them on the range $[-\pi, \pi]$.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x) \sin kx dx &= b_k \int_{-\pi}^{\pi} \sin^2 kx dx \\
&= b_k \pi
\end{aligned}$$

Then we calculate b_k as follows.

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin kx dx \\ &= 0 \quad (\text{since } x^2 \sin kx \text{ is an odd function}) \end{aligned}$$

So the linear combination that is closest to the function $f(x)$ is

$$\frac{\pi^2}{3} + \sum_{k=1}^n \frac{4}{k^2} (-1)^k \cos kx.$$

The Fourier expansion of $f(x) = x^2$ is the limit of the above linear combination as n goes to infinity:

$$\frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2} (-1)^k \cos kx.$$

Supplement We depict the partial summation of this series to the term of $\cos 5x$

$$\frac{\pi^2}{3} + \sum_{k=1}^5 \frac{4}{k^2} (-1)^k \cos kx = \frac{\pi^2}{3} - 4 \cos x + \cos 2x - \frac{4}{9} \cos 3x + \frac{1}{4} \cos 4x - \frac{4}{25} \cos 5x$$

and $f(x) = x^2$ in Fig. 1.

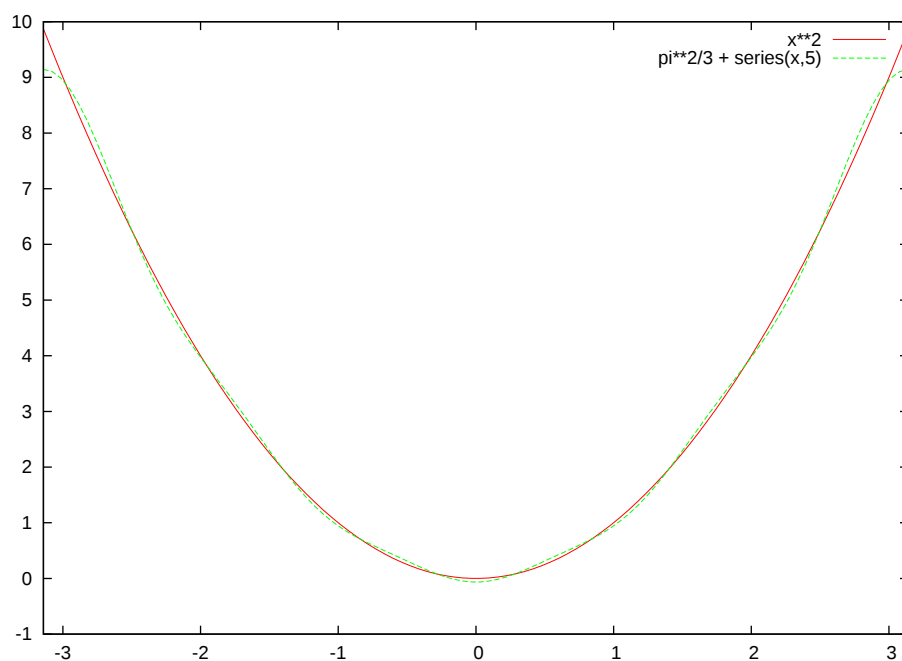


Figure 1: Comparison between the function $f(x) = x^2$ and the partial sum up to the term of $\cos 5x$