

# Exercise 3

Isao Sasano

April 19, 2016

**Exercise 3** Fit a parabola (a square function) to the function  $\cos x$  on  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  so that (the half of) the integral of the squares of the distances between them, where the distance is measured in the vertical direction (the y-direction).

**Solution** Let the function be  $f(x) = ax^2 + bx + c$ . The half of the integral of the squares of the distances between  $f(x)$  and  $\cos x$  on  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  is given as follows.

$$\begin{aligned} J &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{f(x) - \cos x\}^2 dx \\ &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 dx \end{aligned}$$

$J$  takes the minimum value in the point where the partial derivatives of  $J$  with respect to  $a$ ,  $b$ , and  $c$  are 0.

$$\frac{\partial J}{\partial a} = 0, \quad \frac{\partial J}{\partial b} = 0, \quad \frac{\partial J}{\partial c} = 0$$

Firstly the partial derivative of  $J$  with respect to  $a$  is calculated as follows.

$$\begin{aligned} \frac{\partial J}{\partial a} &= \frac{\partial}{\partial a} \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 dx \\ &= \frac{1}{2} \frac{\partial}{\partial a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 dx \\ &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial}{\partial a} \{ax^2 + bx + c - \cos x\}^2 dx \\ &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\{ax^2 + bx + c - \cos x\}x^2 dx \end{aligned}$$

$$\begin{aligned}
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}x^2 dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^4 + bx^3 + cx^2 - x^2 \cos x\} dx \\
&= a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^4 dx + b \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 dx + c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \cos x dx
\end{aligned}$$

Here we calculate each of the integrals. As for  $x^4$  we obtain

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^4 dx &= 2 \int_0^{\frac{\pi}{2}} x^4 dx \quad (\text{since } x^4 \text{ is an even function}) \\
&= 2 \left[ \frac{x^5}{5} \right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi^5}{80}
\end{aligned}$$

As for  $x^3$  its integral is 0 since it is an odd function.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 dx = 0$$

As for  $x^2$  we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 dx = 2 \int_0^{\frac{\pi}{2}} x^2 dx = 2 \left[ \frac{x^3}{3} \right]_0^{\frac{\pi}{2}} = 2 \cdot \frac{\pi^3}{24} = \frac{\pi^3}{12}$$

As for  $x^2 \cos x$  we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \cos x dx = 2 \int_0^{\frac{\pi}{2}} x^2 \cos x dx \quad (\text{since } x^2 \cos x \text{ is an even function})$$

In the following we calculate  $\int_0^{\frac{\pi}{2}} x^2 \cos x dx$ .

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} x^2 \cos x dx &= \left[ x^2 \sin x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2x \sin x dx \\
&= \frac{\pi^2}{4} - 2 \int_0^{\frac{\pi}{2}} x \sin x dx
\end{aligned}$$

Here we calculate  $\int_0^{\frac{\pi}{2}} x \sin x dx$ .

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} x \sin x dx &= \left[ x \frac{\cos x}{-1} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\cos x}{-1} dx \\
&= [\sin x]_0^{\frac{\pi}{2}} \\
&= 1
\end{aligned}$$

Now we resume the calculation of  $\int_0^{\frac{\pi}{2}} x^2 \cos x dx$ .

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x^2 \cos x dx &= \frac{\pi^2}{4} - 2 \cdot 1 \\ &= \frac{\pi^2}{4} - 2\end{aligned}$$

Thus  $\frac{\partial J}{\partial a}$  is obtained as follows.

$$\begin{aligned}\frac{\partial J}{\partial a} &= \frac{\pi^5}{80}a + \frac{\pi^3}{12}c - 2\left(\frac{\pi^2}{4} - 2\right) \\ &= \frac{\pi^5}{80}a + \frac{\pi^3}{12}c - \frac{\pi^2}{2} + 4\end{aligned}$$

Secondly the partial derivative of  $J$  with respect to  $b$  is calculated as follows.

$$\begin{aligned}\frac{\partial J}{\partial b} &= \frac{\partial}{\partial b} \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 dx \\ &= \frac{1}{2} \frac{\partial}{\partial b} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 dx \\ &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial}{\partial b} \{ax^2 + bx + c - \cos x\}^2 dx \\ &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\{ax^2 + bx + c - \cos x\}x dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}x dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^3 + bx^2 + cx - x \cos x\} dx \\ &= a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 dx + b \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 dx + c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos x dx \\ &= 2b \int_0^{\frac{\pi}{2}} x^2 dx \quad (x^3, x, \text{ and } x \cos x \text{ are odd functions and } x^2 \text{ is an even function}) \\ &= 2b \cdot \frac{\pi^3}{24} \\ &= \frac{\pi^3}{12}b\end{aligned}$$

Thirdly the partial derivative of  $J$  with respect to  $c$  is calculated as follows.

$$\frac{\partial J}{\partial c} = \frac{\partial}{\partial c} \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 dx$$

$$\begin{aligned}
&= \frac{1}{2} \frac{\partial}{\partial c} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 dx \\
&= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial}{\partial c} \{ax^2 + bx + c - \cos x\}^2 dx \\
&= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\{ax^2 + bx + c - \cos x\} \cdot 1 dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\} dx \\
&= a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 dx + b \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x dx + c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx \\
&= 2a \int_0^{\frac{\pi}{2}} x^2 dx + \pi c - 2 \int_0^{\frac{\pi}{2}} \cos x dx \\
&= 2a \cdot \frac{\pi^3}{24} + \pi c - 2 [\sin x]_0^{\frac{\pi}{2}} \\
&= \frac{\pi^3}{12} a + \pi c - 2
\end{aligned}$$

Thus we obtain the system of equations with respect to  $a$ ,  $b$ , and  $c$ .

$$\begin{aligned}
\frac{\pi^5}{80} a + \frac{\pi^3}{12} c - \frac{\pi^2}{2} + 4 &= 0 \\
\frac{\pi^3}{12} b &= 0 \\
\frac{\pi^3}{12} a + \pi c - 2 &= 0
\end{aligned}$$

By solving this, we obtain the solution.

$$a = \frac{60\pi^2 - 720}{\pi^5}, \quad b = 0, \quad c = \frac{60 - 3\pi^2}{\pi^3}$$

Hence the function is obtained as follows.

$$f(x) = \frac{60\pi^2 - 720}{\pi^5} x^2 + \frac{60 - 3\pi^2}{\pi^3}$$

The function is depicted with  $\cos x$  on  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  in Fig. 1. In Fig. 1 the red curve is the square function and the green curve is the function  $\cos x$ .

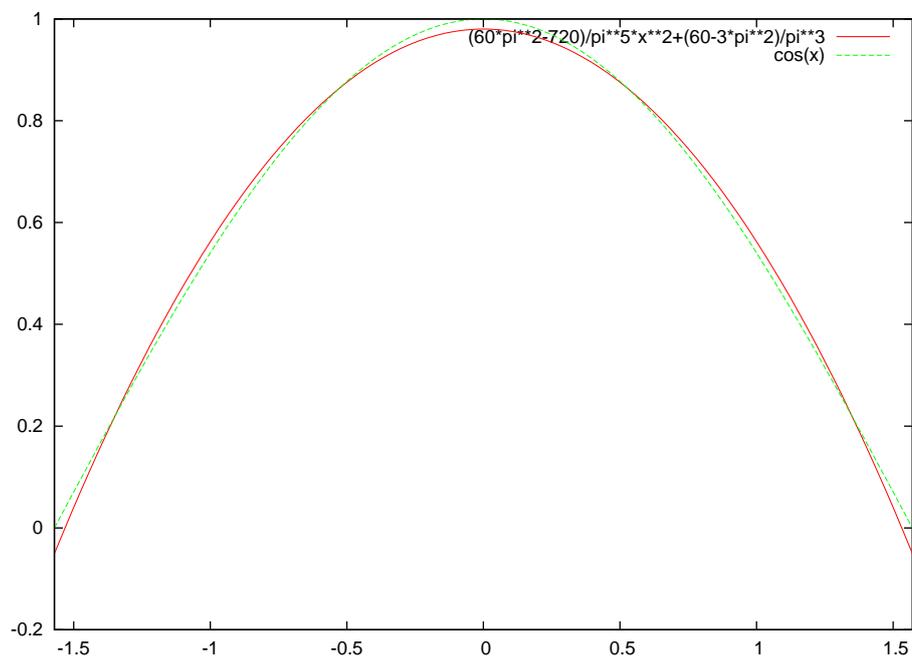


Figure 1: The closest square function to  $\cos x$  on the range  $[-\frac{\pi}{2}, \frac{\pi}{2}]$