

Exercise 14-1

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Exercise Calculate the Fourier transform of the following function $f(x)$.

$$f(x) = \begin{cases} x & 0 < x < L \\ 0 & \text{otherwise} \end{cases}$$

Here we define the Fourier transform and the inverse Fourier transform as follows¹.

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega \\ F(\omega) &= \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \end{aligned}$$

The above equations are definitions of transformations. Note that the inverse Fourier transform of the Fourier transform of $f(x)$ is not necessarily equal to $f(x)$.

¹In the definitions the constant factor depends on textbooks.

Solution

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx \\ &= \int_0^L xe^{-i\omega x} dx \\ &= \left[x \frac{e^{-i\omega x}}{-i\omega} \right]_0^L - \int_0^L \frac{e^{-i\omega x}}{-i\omega} dx \\ &= \frac{Le^{-i\omega L}}{-i\omega} - \left[\frac{e^{-i\omega x}}{-\omega^2} \right]_0^L \\ &= \frac{Le^{-i\omega L}}{-i\omega} + \left[\frac{e^{-i\omega x}}{\omega^2} \right]_0^L \\ &= i \frac{Le^{-i\omega L}}{\omega} + \frac{e^{-i\omega L} - 1}{\omega^2} \\ &= i \frac{\omega Le^{-i\omega L}}{\omega^2} + \frac{e^{-i\omega L} - 1}{\omega^2} \\ &= \frac{i\omega Le^{-i\omega L} + e^{-i\omega L} - 1}{\omega^2} \\ &= \frac{e^{-i\omega L}(1 + i\omega L) - 1}{\omega^2} \\ &= \frac{e^{-iL\omega}(1 + iL\omega) - 1}{\omega^2} \end{aligned}$$

Comment The inverse Fourier transform of $F(\omega)$ obtained above is as follows.

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iL\omega}(1 + iL\omega) - 1}{\omega^2} e^{i\omega x} d\omega = \begin{cases} x & 0 < x < L \\ L/2 & x = L \\ 0 & \text{otherwise} \end{cases}$$

Note that I do not obtain the inverse Fourier transform by calculating the above integral but obtained it from $f(x)$. In general, the following equation holds, similarly to the one in the Fourier series.

$$\mathcal{F}^{-1}(\mathcal{F}(f))(x) = \frac{f(x+0) + f(x-0)}{2}$$