# Exercise 12

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**Exercise** Consider the set of real-valued continuous functions on the interval $[-1, 1]$. As we did in Exercise 9-1, we construct an inner product space where the addition, scalar multiplication, and inner product are defined as follows.

$$\begin{aligned}(\boldsymbol{f}+\boldsymbol{g})(x) &= f(x)+g(x)\\ (c\boldsymbol{f})(x) &= c(f(x))\\ (\boldsymbol{f},\boldsymbol{g}) &= \int_{-1}^{1} f(x)g(x)\mathrm{d}x\end{aligned}$$

On this inner product space, apply the Gram-Schmidt orthogonalization to the four vectors (functions) $u_1(x) = 1,\ u_2(x) = x,\ u_3(x) = x^2, u_4(x) = x^3$.

**Solution** Let $e_1 = u_1$. So $e_1(x) = 1$.

We let $e_2 = u_2 - c_1e_1$ and calculate $c_1$ so that $e_2$ is orthogonal to $e_1$. The inner product of $e_1$ and $e_2$ is calculated as follows.

$$\begin{aligned}(e_1, e_2) &= (e_1, u_2 - c_1e_1)\\ &= (e_1, u_2) - c_1(e_1, e_1)\end{aligned}$$

We let this be 0. Then we obtain $c_1$ as follows.

$$\begin{aligned}c_1 &= \frac{(e_1, u_2)}{(e_1, e_1)}\\ &= \frac{(1, x)}{(1, 1)}\\ &= \frac{\int_{-1}^{1} x\mathrm{d}x}{\int_{-1}^{1} 1\mathrm{d}x}\\ &= 0\end{aligned}$$

So we obtain $e_2$ as follows.

$$e_2(x) = x$$

Next we let $e_3 = u_3 - (c_1 e_1 + c_2 e_2)$ and calculate $c_1$ and $c_2$ so that $e_3$ is orthogonal to $e_1$ and $e_2$. The inner product of $e_1$ and $e_3$ is calculated as follows.

$$\begin{aligned}(e_1, e_3) &= (e_1, u_3 - (c_1 e_1 + c_2 e_2)) \\ &= (e_1, u_3) - c_1(e_1, e_1)\end{aligned}$$

We let this be 0. Then we obtain $c_1$ as follows.

$$\begin{aligned}c_1 &= \frac{(e_1, u_3)}{(e_1, e_1)} \\ &= \frac{(1, x^2)}{(1, 1)} \\ &= \frac{\int_{-1}^{1} x^2 \mathrm{d}x}{\int_{-1}^{1} 1 \mathrm{d}x} \\ &= \frac{2/3}{2} \\ &= \frac{1}{3}\end{aligned}$$

The inner product of $e_2$ and $e_3$ is calculated as follows.

$$\begin{aligned}(e_2, e_3) &= (e_2, u_3 - (c_1 e_1 + c_2 e_2)) \\ &= (e_2, u_3) - c_2(e_2, e_2)\end{aligned}$$

We let this be 0. Then we obtain $c_2$ as follows.

$$\begin{aligned}c_2 &= \frac{(e_2, u_3)}{(e_2, e_2)} \\ &= \frac{(x, x^2)}{(x, x)} \\ &= \frac{\int_{-1}^{1} x^3 \mathrm{d}x}{\int_{-1}^{1} x^2 \mathrm{d}x} \\ &= 0\end{aligned}$$

So we obtain $e_3$ as follows.

$$e_3(x) = x^2 - \frac{1}{3}$$

Finally we let $e_4 = u_4 - (c_1 e_1 + c_2 e_2 + c_3 e_3)$ and calculate $c_1$, $c_2$, and $c_3$ so that $e_4$ is orthogonal to $e_1$, $e_2$, and $e_3$. The inner product of $e_1$ and $e_4$ is calculated as follows.

$$\begin{aligned} (e_1, e_4) &= (e_1, u_4 - (c_1 e_1 + c_2 e_2 + c_3 e_3)) \\ &= (e_1, u_4) - c_1 (e_1, e_1) \end{aligned}$$

We let this be 0. Then we obtain $c_1$ as follows.

$$\begin{aligned} c_1 &= \frac{(e_1, u_4)}{(e_1, e_1)} \\ &= \frac{(1, x^3)}{(1, 1)} \\ &= \frac{\int_{-1}^{1} x^3 \mathrm{d}x}{\int_{-1}^{1} 1 \mathrm{d}x} \\ &= 0 \end{aligned}$$

The inner product of $e_2$ and $e_4$ is calculated as follows.

$$\begin{aligned} (e_2, e_4) &= (e_2, u_4 - (c_1 e_1 + c_2 e_2 + c_3 e_3)) \\ &= (e_2, u_4) - c_2 (e_2, e_2) \end{aligned}$$

We let this be 0. Then we obtain $c_2$ as follows.

$$\begin{aligned} c_2 &= \frac{(e_2, u_4)}{(e_2, e_2)} \\ &= \frac{(x, x^3)}{(x, x)} \\ &= \frac{\int_{-1}^{1} x^4 \mathrm{d}x}{\int_{-1}^{1} x^2 \mathrm{d}x} \\ &= \frac{2/5}{2/3} \\ &= \frac{3}{5} \end{aligned}$$

The inner product of $e_3$ and $e_4$ is calculated as follows.

$$\begin{aligned}(e_3, e_4) &= (e_3, u_4 - (c_1e_1 + c_2e_2 + c_3e_3)) \\ &= (e_3, u_4) - c_3(e_3, e_3)\end{aligned}$$

We let this be 0. Then we obtain $c_3$ as follows.

$$\begin{aligned}c_3 &= \frac{(e_3, u_4)}{(e_3, e_3)} \\ &= \frac{(x^2 - \frac{1}{3}, x^3)}{(x^2 - \frac{1}{3}, x^2 - \frac{1}{3})} \\ &= \frac{\int_{-1}^{1} x^5 - \frac{1}{3}x^3 \mathrm{d}x}{\int_{-1}^{1} x^4 - \frac{2}{3}x^2 + \frac{1}{9}\mathrm{d}x} \\ &= 0\end{aligned}$$

So we obtain $e_4$ as follows.

$$e_4(x) = x^3 - \frac{3}{5}x$$

So the orthogonal vectors (functions) obtained from $1, x, x^2, x^3$ by the Gram-Schmidt orthogonalization are as follows.

$$1,\ x,\ x^2 - \frac{1}{3},\ x^3 - \frac{3}{5}x$$

**Note1** The resulting vectors (functions) are not normalized. Of course we may normalize them.
**Note2** The obtained vectors (functions) are related to the Legendre polynomials with the following relationship.

$$P_i(x) = \frac{e_{i+1}(x)}{e_{i+1}(1)} \quad (i \geq 0)$$

We check the cases for $i = 0, 1, 2, 3$.

$$\begin{aligned}\frac{e_1(x)}{e_1(1)} &= \frac{1}{1} \\ &= 1 \\ &= P_0(x)\end{aligned}$$

$$
\begin{aligned}
\frac{e_2(x)}{e_2(1)} &= \frac{x}{1} \\
&= x \\
&= P_1(x)
\end{aligned}
$$

$$
\begin{aligned}
\frac{e_3(x)}{e_3(1)} &= \frac{x^2 - \frac{1}{3}}{\frac{2}{3}} \\
&= (x^2 - \frac{1}{3}) \cdot \frac{3}{2} \\
&= \frac{1}{2}(3x^2 - 1) \\
&= P_2(x)
\end{aligned}
$$

$$
\begin{aligned}
\frac{e_4(x)}{e_4(1)} &= \frac{x^3 - \frac{3}{5}x}{1 - \frac{3}{5}} \\
&= \frac{x^3 - \frac{3}{5}x}{\frac{2}{5}} \\
&= (x^3 - \frac{3}{5}x) \cdot \frac{5}{2} \\
&= \frac{1}{2}(5x^3 - 3x) \\
&= P_3(x)
\end{aligned}
$$