

Three solutions for Exercise 10-2

Isao Sasano

2016 June 21

Exercise Consider the set of real-valued continuous functions on the interval $[-\pi, \pi]$. As we did in Exercise 9-1, we construct an inner product space where the addition, scalar multiplication, and inner product are defined as follows.

$$\begin{aligned}(\mathbf{f} + \mathbf{g})(x) &= f(x) + g(x) \\(c\mathbf{f})(x) &= c(f(x)) \\(\mathbf{f}, \mathbf{g}) &= \int_{-\pi}^{\pi} f(x)g(x)dx\end{aligned}$$

On this inner product space, approximate a function $f(x) = x^2$ by a linear combination of the functions $e_1(x) = \frac{1}{2}$, $e_2(x) = \cos x$, and $e_3(x) = \sin x$ (i.e., $\sum_{k=1}^3 c_k e_k(x) = c_1 e_1(x) + c_2 e_2(x) + c_3 e_3(x)$ for some c_1 , c_2 , and c_3). That is, obtain c_1 , c_2 , c_3 so that $c_1 e_1(x) + c_2 e_2(x) + c_3 e_3(x)$ is closest to $f(x)$. As for the measure of the distance, use (the half of) the square of the norm of the difference of $c_1 e_1(x) + c_2 e_2(x) + c_3 e_3(x)$ and $f(x)$.

$$J = \frac{1}{2} \left\| \sum_{k=1}^3 c_k \mathbf{e}_k - \mathbf{f} \right\|^2$$

The norm is defined as follows.

$$\|\mathbf{f}\| = \sqrt{(\mathbf{f}, \mathbf{f})} = \sqrt{\int_{-\pi}^{\pi} f(x)^2 dx}$$

We show three solutions. One is by solving $\frac{\partial J}{\partial c_i} = 0$ for $i = 1, 2, 3$. Another is by considering the special case. The other is by calculating the projection of \mathbf{f} on the subspace spanned by \mathbf{e}_1 , \mathbf{e}_2 , and \mathbf{e}_3 .

Solution 1 Firstly calculate J as follows.

$$\begin{aligned}
 J &= \frac{1}{2} \left\| \sum_{k=1}^3 c_k \mathbf{e}_k - \mathbf{f} \right\|^2 \\
 &= \frac{1}{2} \left(\sum_{k=1}^3 c_k \mathbf{e}_k - \mathbf{f}, \sum_{k=1}^3 c_k \mathbf{e}_k - \mathbf{f} \right) \\
 &= \frac{1}{2} \left\{ \left(\sum_{k=1}^3 c_k \mathbf{e}_k, \sum_{k=1}^3 c_k \mathbf{e}_k \right) - 2 \left(\mathbf{f}, \sum_{k=1}^3 c_k \mathbf{e}_k \right) + \|\mathbf{f}\|^2 \right\} \\
 &= \frac{1}{2} \left\{ \sum_{k,l=1}^3 c_k c_l (\mathbf{e}_k, \mathbf{e}_l) - 2 \sum_{k=1}^3 c_k (\mathbf{f}, \mathbf{e}_k) + \|\mathbf{f}\|^2 \right\}
 \end{aligned}$$

Partially differentiate this with respect to c_i ($i = 1, 2, 3$).

$$\begin{aligned}
 \frac{\partial J}{\partial c_i} &= \frac{\partial}{\partial c_i} \frac{1}{2} \left\{ \sum_{k,l=1}^3 c_k c_l (\mathbf{e}_k, \mathbf{e}_l) - 2 \sum_{k=1}^3 c_k (\mathbf{f}, \mathbf{e}_k) + \|\mathbf{f}\|^2 \right\} \\
 &= \frac{1}{2} \left\{ \frac{\partial}{\partial c_i} \sum_{k,l=1}^3 c_k c_l (\mathbf{e}_k, \mathbf{e}_l) - 2 \frac{\partial}{\partial c_i} \sum_{k=1}^3 c_k (\mathbf{f}, \mathbf{e}_k) \right\} \\
 &= \frac{1}{2} \left\{ 2 \sum_{k=1}^3 c_k (\mathbf{e}_k, \mathbf{e}_i) - 2 (\mathbf{f}, \mathbf{e}_i) \right\} \\
 &= \sum_{k=1}^3 c_k (\mathbf{e}_k, \mathbf{e}_i) - (\mathbf{f}, \mathbf{e}_i)
 \end{aligned}$$

By writing $\frac{\partial J}{\partial c_i} = 0$ for $i = 1, 2, 3$ in matrix form we obtain

$$\begin{pmatrix} (\mathbf{e}_1, \mathbf{e}_1) & (\mathbf{e}_2, \mathbf{e}_1) & (\mathbf{e}_3, \mathbf{e}_1) \\ (\mathbf{e}_1, \mathbf{e}_2) & (\mathbf{e}_2, \mathbf{e}_2) & (\mathbf{e}_3, \mathbf{e}_2) \\ (\mathbf{e}_1, \mathbf{e}_3) & (\mathbf{e}_2, \mathbf{e}_3) & (\mathbf{e}_3, \mathbf{e}_3) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} (\mathbf{f}, \mathbf{e}_1) \\ (\mathbf{f}, \mathbf{e}_2) \\ (\mathbf{f}, \mathbf{e}_3) \end{pmatrix}$$

Since \mathbf{e}_1 , \mathbf{e}_2 , and \mathbf{e}_3 are orthogonal to each other, we obtain

$$\begin{pmatrix} \|\mathbf{e}_1\|^2 & 0 & 0 \\ 0 & \|\mathbf{e}_2\|^2 & 0 \\ 0 & 0 & \|\mathbf{e}_3\|^2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} (\mathbf{f}, \mathbf{e}_1) \\ (\mathbf{f}, \mathbf{e}_2) \\ (\mathbf{f}, \mathbf{e}_3) \end{pmatrix}$$

Hence we obtain $c_i = \frac{(\mathbf{f}, \mathbf{e}_i)}{\|\mathbf{e}_i\|^2}$ for $i = 1, 2, 3$.

$$\begin{aligned}
 (\mathbf{f}, \mathbf{e}_1) &= \int_{-\pi}^{\pi} \frac{x^2}{2} dx = \left[\frac{x^3}{6} \right]_{-\pi}^{\pi} = \frac{\pi^3}{3} \\
 (\mathbf{f}, \mathbf{e}_2) &= \int_{-\pi}^{\pi} x^2 \cos x dx \\
 &= \left[x^2 \frac{\sin x}{-1} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \frac{\sin x}{-1} dx \\
 &= 2 \int_{-\pi}^{\pi} x \sin x dx \\
 &= 2 \left\{ [x \cos x]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \cos x dx \right\} \\
 &= 2 \left\{ \pi \cos \pi - (-\pi) \cos(-\pi) \right\} \\
 &= 2 \{ 2\pi \cos \pi \} \\
 &= 4\pi \cos \pi \\
 &= -4\pi \\
 (\mathbf{f}, \mathbf{e}_3) &= \int_{-\pi}^{\pi} x^2 \sin x dx = 0 \\
 (\mathbf{e}_1, \mathbf{e}_1) &= \int_{-\pi}^{\pi} \frac{1}{4} dx = \left[\frac{x}{4} \right]_{-\pi}^{\pi} = \frac{\pi}{2} \\
 (\mathbf{e}_2, \mathbf{e}_2) &= \int_{-\pi}^{\pi} \cos^2 x dx = \pi
 \end{aligned}$$

So we obtain

$$\begin{aligned}
 c_1 &= \frac{\frac{\pi^3}{3}}{\frac{\pi}{2}} = \frac{2}{3}\pi^2 \\
 c_2 &= \frac{-4\pi}{\pi} = -4 \\
 c_3 &= 0
 \end{aligned}$$

Thus the linear combination is obtained as follows.

$$\frac{2}{3}\pi^2 \mathbf{e}_1 - 4\mathbf{e}_2$$

This vector represents the following function.

$$\left(\frac{2}{3}\pi^2 \mathbf{e}_1 - 4\mathbf{e}_2 \right)(x) = \frac{2}{3}\pi^2 e_1(x) - 4e_2(x)$$

$$= \frac{1}{3}\pi^2 - 4\cos x$$

Notice that the function consists of the terms up to the terms of $\cos x$ and $\sin x$ in the Fourier series of $f(x)$.

Solution 2 Assume the following equation holds.

$$f = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$$

(There is actually no coefficients c_1 , c_2 , and c_3 that satisfy the equation, but it's ok.) Take the inner product with \mathbf{e}_1 , \mathbf{e}_2 , and \mathbf{e}_3 in the both side.

$$\begin{aligned}(\mathbf{f}, \mathbf{e}_1) &= c_1(\mathbf{e}_1, \mathbf{e}_1) + c_2(\mathbf{e}_2, \mathbf{e}_1) + c_3(\mathbf{e}_3, \mathbf{e}_1) \\(\mathbf{f}, \mathbf{e}_2) &= c_1(\mathbf{e}_1, \mathbf{e}_2) + c_2(\mathbf{e}_2, \mathbf{e}_2) + c_3(\mathbf{e}_3, \mathbf{e}_2) \\(\mathbf{f}, \mathbf{e}_3) &= c_1(\mathbf{e}_1, \mathbf{e}_3) + c_2(\mathbf{e}_2, \mathbf{e}_3) + c_3(\mathbf{e}_3, \mathbf{e}_3)\end{aligned}$$

We write the above three equations in the matrix form.

$$\begin{pmatrix} (\mathbf{e}_1, \mathbf{e}_1) & (\mathbf{e}_2, \mathbf{e}_1) & (\mathbf{e}_3, \mathbf{e}_1) \\ (\mathbf{e}_1, \mathbf{e}_2) & (\mathbf{e}_2, \mathbf{e}_2) & (\mathbf{e}_3, \mathbf{e}_2) \\ (\mathbf{e}_1, \mathbf{e}_3) & (\mathbf{e}_2, \mathbf{e}_3) & (\mathbf{e}_3, \mathbf{e}_3) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} (\mathbf{f}, \mathbf{e}_1) \\ (\mathbf{f}, \mathbf{e}_2) \\ (\mathbf{f}, \mathbf{e}_3) \end{pmatrix}$$

Notice that this is the same as the one obtained in Solution 1. So we omit the remaining calculation.

Solution 3

A linear combination of \mathbf{e}_1 , \mathbf{e}_2 , and \mathbf{e}_3 is written as $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$. It is closest to f when it is the projection of f on the subspace spanned by \mathbf{e}_1 , \mathbf{e}_2 , and \mathbf{e}_3 . That is, the vector $f - (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3)$ is orthogonal to the subspace spanned by \mathbf{e}_1 , \mathbf{e}_2 , and \mathbf{e}_3 . So the following three equations should hold.

$$\begin{aligned}(f - (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3), \mathbf{e}_1) &= 0 \\(f - (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3), \mathbf{e}_2) &= 0 \\(f - (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3), \mathbf{e}_3) &= 0\end{aligned}$$

We expand the inner product in each of the equations.

$$\begin{aligned}(\mathbf{f}, \mathbf{e}_1) - (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3, \mathbf{e}_1) &= 0 \\(\mathbf{f}, \mathbf{e}_2) - (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3, \mathbf{e}_2) &= 0 \\(\mathbf{f}, \mathbf{e}_3) - (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3, \mathbf{e}_3) &= 0\end{aligned}$$

We move the second inner product into RHS in each of the equations.

$$\begin{aligned}(\mathbf{f}, \mathbf{e}_1) &= (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3, \mathbf{e}_1) \\(\mathbf{f}, \mathbf{e}_2) &= (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3, \mathbf{e}_2) \\(\mathbf{f}, \mathbf{e}_3) &= (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3, \mathbf{e}_3)\end{aligned}$$

We expand the inner product in the RHS in each of the equations.

$$\begin{aligned}(\mathbf{f}, \mathbf{e}_1) &= c_1(\mathbf{e}_1, \mathbf{e}_1) + c_2(\mathbf{e}_2, \mathbf{e}_1) + c_3(\mathbf{e}_3, \mathbf{e}_1) \\(\mathbf{f}, \mathbf{e}_2) &= c_1(\mathbf{e}_1, \mathbf{e}_2) + c_2(\mathbf{e}_2, \mathbf{e}_2) + c_3(\mathbf{e}_3, \mathbf{e}_2) \\(\mathbf{f}, \mathbf{e}_3) &= c_1(\mathbf{e}_1, \mathbf{e}_3) + c_2(\mathbf{e}_2, \mathbf{e}_3) + c_3(\mathbf{e}_3, \mathbf{e}_3)\end{aligned}$$

We write the above three equations in the matrix form.

$$\begin{pmatrix} (\mathbf{e}_1, \mathbf{e}_1) & (\mathbf{e}_2, \mathbf{e}_1) & (\mathbf{e}_3, \mathbf{e}_1) \\ (\mathbf{e}_1, \mathbf{e}_2) & (\mathbf{e}_2, \mathbf{e}_2) & (\mathbf{e}_3, \mathbf{e}_2) \\ (\mathbf{e}_1, \mathbf{e}_3) & (\mathbf{e}_2, \mathbf{e}_3) & (\mathbf{e}_3, \mathbf{e}_3) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} (\mathbf{f}, \mathbf{e}_1) \\ (\mathbf{f}, \mathbf{e}_2) \\ (\mathbf{f}, \mathbf{e}_3) \end{pmatrix}$$

Notice that this is the same as the one obtained in Solution 1 and 2. So we omit the remaining calculation.