Solutions for Mid-term examination

Isao Sasano

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Problem 1 (10 points) Fit a straight line (a linear function) to the three points $(0,0),(1,1),(3,4)$ so that (the half of) the sum of the squares of the distances of those points from the straight line is minimum, where the distance is measured in the vertical direction (the y-direction).

Solution Let the line (the linear function) be $f(x) = ax+b$ and $(x_1,y_1) = (0,0),(x_2,y_2) = (1,1),(x_3,y_3) = (3,4)$. The half of the sum of the squares of the distances of these points from the line is given as follows.

$$J = \frac{1}{2}\sum_{i=1}^{3}(f(x_i) - y_i)^2 = \frac{1}{2}\sum_{i=1}^{3}(ax_i + b - y_i)^2$$

J takes the minimum value in the point where the partial derivatives of J with respect to a and b are 0.

$$\frac{\partial J}{\partial a} = 0, \frac{\partial J}{\partial b} = 0$$

Firstly the partial derivative of J with respect to a is calculated as follows.

$$\begin{aligned}
\frac{\partial J}{\partial a} &= \frac{\partial}{\partial a}\{\frac{1}{2}\sum_{i=1}^{3}(ax_i + b - y_i)^2\} \\
&= \frac{1}{2}\sum_{i=1}^{3}\frac{\partial}{\partial a}(ax_i + b - y_i)^2 \\
&= \frac{1}{2}\sum_{i=1}^{3}2(ax_i + b - y_i)x_i \\
&= \sum_{i=1}^{3}(ax_i + b - y_i)x_i \\
&= \sum_{i=1}^{3}(ax_i^2 + bx_i - x_iy_i) \\
&= a\sum_{i=1}^{3}x_i^2 + b\sum_{i=1}^{3}x_i - \sum_{i=1}^{3}x_iy_i
\end{aligned}$$

Secondly the partial derivative of J with respect to b is calculated as follows.

$$\frac{\partial J}{\partial b} = \frac{\partial}{\partial b}\{\frac{1}{2}\sum_{i=1}^{3}(ax_i + b - y_i)^2\}$$

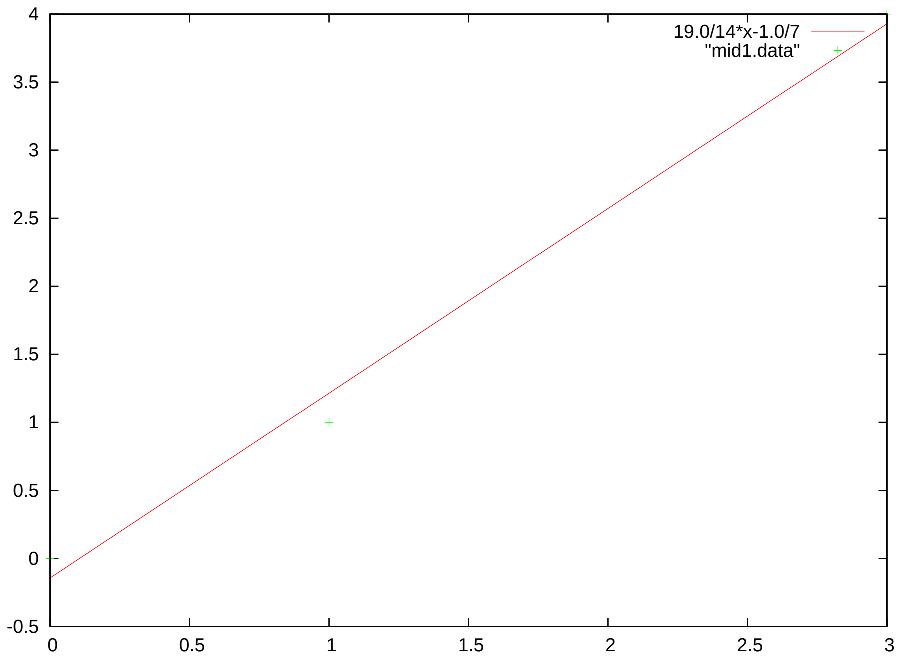

Figure 1: The straight line closest to the given three points

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{3}\frac{\partial}{\partial b}(ax_i+b-y_i)^2 \\
&= \frac{1}{2}\sum_{i=1}^{3}2(ax_i+b-y_i) \\
&= \sum_{i=1}^{3}(ax_i+b-y_i) \\
&= a\sum_{i=1}^{3}x_i+b\sum_{i=1}^{3}1-\sum_{i=1}^{3}y_i
\end{aligned}
$$

Then we obtain the system of equations

$$
\begin{aligned}
10a+4b-13 &= 0 \\
4a+3b-5 &= 0
\end{aligned}
$$

and $a = \frac{19}{14}, b = -\frac{1}{7}$ is the solution. Hence the function is obtained as follows.

$$f(x) = \frac{19}{14}x - \frac{1}{7}$$

Supplement: The function is depicted with the three points in Fig. 1.

Problem 2 (10 points) Fit a parabola (a square function) to the four points $(-1,0),(0,-1),(1,0),(2,1)$ so that (the half of) the sum of the squares of the distances of those points from the parabola is minimum, where the distance is measured in the vertical direction (the y-direction).

Solution Let the function be $f(x) = ax^2+bx+c$ and $(x_1,y_1) = (-1,0),(x_2,y_2) = (0,-1),(x_3,y_3) = (1,0),(x_4,y_4) = (2,1)$. The half of the sum of the squares of the distances of these points from the parabola is given as follows.

$$J = \frac{1}{2}\sum_{i=1}^{4}(f(x_i) - y_i)^2 = \frac{1}{2}\sum_{i=1}^{4}(ax_i^2 + bx_i + c - y_i)^2$$

J takes the minimum value in the point where the partial derivatives of J with respect to a, b, and c are 0.

$$\frac{\partial J}{\partial a} = 0, \quad \frac{\partial J}{\partial b} = 0, \quad \frac{\partial J}{\partial c} = 0$$

Firstly the partial derivative of J with respect to a is calculated as follows.

$$\begin{aligned}
\frac{\partial J}{\partial a} &= \frac{\partial}{\partial a}\{\frac{1}{2}\sum_{i=1}^{4}(ax_i^2 + bx_i + c - y_i)^2\} \\
&= \frac{1}{2}\sum_{i=1}^{4}\frac{\partial}{\partial a}(ax_i^2 + bx_i + c - y_i)^2 \\
&= \frac{1}{2}\sum_{i=1}^{4}2(ax_i^2 + bx_i + c - y_i)\frac{\partial}{\partial a}(ax_i^2 + bx_i + c - y_i) \\
&= \frac{1}{2}\sum_{i=1}^{4}2(ax_i^2 + bx_i + c - y_i)x_i^2 \\
&= \sum_{i=1}^{4}(ax_i^2 + bx_i + c - y_i)x_i^2 \\
&= \sum_{i=1}^{4}(ax_i^4 + bx_i^3 + cx_i^2 - x_i^2 y_i) \\
&= a\sum_{i=1}^{4}x_i^4 + b\sum_{i=1}^{4}x_i^3 + c\sum_{i=1}^{4}x_i^2 - \sum_{i=1}^{4}x_i^2 y_i
\end{aligned}$$

Secondly the partial derivative of J with respect to b is calculated as follows.

$$\begin{aligned}
\frac{\partial J}{\partial b} &= \frac{\partial}{\partial b}\{\frac{1}{2}\sum_{i=1}^{4}(ax_i^2 + bx_i + c - y_i)^2\} \\
&= \frac{1}{2}\sum_{i=1}^{4}\frac{\partial}{\partial b}(ax_i^2 + bx_i + c - y_i)^2 \\
&= \frac{1}{2}\sum_{i=1}^{4}2(ax_i^2 + bx_i + c - y_i)\frac{\partial}{\partial b}(ax_i^2 + bx_i + c - y_i) \\
&= \frac{1}{2}\sum_{i=1}^{4}2(ax_i^2 + bx_i + c - y_i)x_i \\
&= \sum_{i=1}^{4}(ax_i^2 + bx_i + c - y_i)x_i \\
&= \sum_{i=1}^{4}(ax_i^3 + bx_i^2 + cx_i - x_i y_i) \\
&= a\sum_{i=1}^{4}x_i^3 + b\sum_{i=1}^{4}x_i^2 + c\sum_{i=1}^{4}x_i - \sum_{i=1}^{4}x_i y_i
\end{aligned}$$

Thirdly the partial derivative of J with respect to c is calculated as follows.

$$\begin{aligned}
\frac{\partial J}{\partial c} &= \frac{\partial}{\partial c}\{\frac{1}{2}\sum_{i=1}^{4}(ax_i^2 + bx_i + c - y_i)^2\} \\
&= \frac{1}{2}\sum_{i=1}^{4}\frac{\partial}{\partial c}(ax_i^2 + bx_i + c - y_i)^2 \\
&= \frac{1}{2}\sum_{i=1}^{4}2(ax_i^2 + bx_i + c - y_i)\frac{\partial}{\partial c}(ax_i^2 + bx_i + c - y_i) \\
&= \frac{1}{2}\sum_{i=1}^{4}2(ax_i^2 + bx_i + c - y_i)1 \\
&= \sum_{i=1}^{4}(ax_i^2 + bx_i + c - y_i) \\
&= a\sum_{i=1}^{4}x_i^2 + b\sum_{i=1}^{4}x_i + c\sum_{i=1}^{4}1 - \sum_{i=1}^{4}y_i
\end{aligned}$$

Then we obtain the system of equations with respect to a, b, and c. The coefficients of the equations are computed as follows.

$$\sum_{i=1}^{4}x_i^4 = 18, \quad \sum_{i=1}^{4}x_i^3 = 8, \quad \sum_{i=1}^{4}x_i^2 = 6, \quad \sum_{i=1}^{4}x_i = 2$$
$$\sum_{i=1}^{4}1 = 4, \quad \sum_{i=1}^{4}x_i^2 y_i = 4, \quad \sum_{i=1}^{4}x_i y_i = 2, \quad \sum_{i=1}^{4}y_i = 0$$

Hence the system of equations is obtained as follows.

$$\begin{aligned}
18a + 8b + 6c - 4 &= 0 \quad \cdots (1) \\
8a + 6b + 2c - 2 &= 0 \quad \cdots (2) \\
6a + 2b + 4c &= 0 \quad \cdots (3)
\end{aligned}$$

By solving this, we obtain the solution.

$$a = \frac{1}{2}, \quad b = -\frac{1}{10}, \quad c = -\frac{7}{10}$$

Hence the function is obtained as follows.

$$f(x) = \frac{1}{2}x^2 - \frac{1}{10}x - \frac{7}{10}$$

Supplement: The function is depicted with the four points in Fig. 2. In Fig. 2 the green symbols are the given points and the red curve is the square function.

Problem 3 (10 points) Approximate a column vector $\boldsymbol{a} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$ by a linear combination of the column vectors $\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

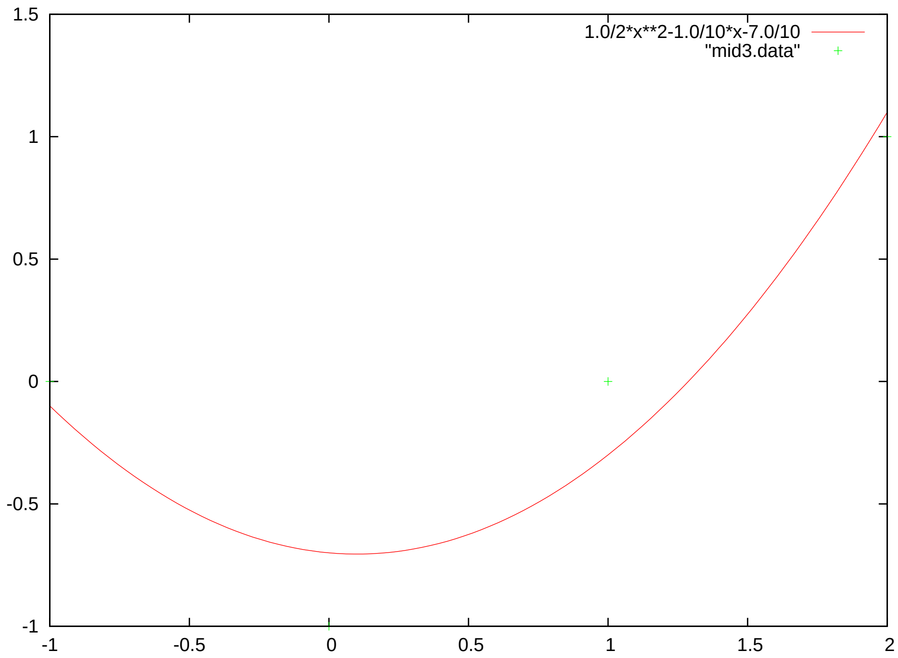

Figure 2: The parabola closest to the given four points

(i.e., $\sum_{k=1}^{2} c_k \boldsymbol{u}_k = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2$ for some c_1 and c_2). As for the measure of the distance, use (the half of) the square of the norm of the difference of $c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2$ and $\boldsymbol{a}$: $J = \frac{1}{2}\left\|\sum_{k=1}^{2} c_k\boldsymbol{u}_k - \boldsymbol{a}\right\|^2$. The norm of a column vector $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is defined to be $\|\boldsymbol{x}\| = \sqrt{(\boldsymbol{x}, \boldsymbol{x})} = \sqrt{\sum_{k=1}^{3} x_k^2}$.

Solutions We show two solutions. One is by substituting the given column vectors into the normal equations and the other is by substituting them from the beginning. Solution 1 is clearer.

Solution 1 Firstly calculate J as follows.

$$\begin{aligned} J &= \frac{1}{2}\left\|\sum_{k=1}^{2} c_k\boldsymbol{u}_k - \boldsymbol{a}\right\|^2 \\ &= \frac{1}{2}\left(\sum_{k=1}^{2} c_k\boldsymbol{u}_k - \boldsymbol{a}, \sum_{k=1}^{2} c_k\boldsymbol{u}_k - \boldsymbol{a}\right) \\ &= \frac{1}{2}\left\{\left(\sum_{k=1}^{2} c_k\boldsymbol{u}_k, \sum_{k=1}^{2} c_k\boldsymbol{u}_k\right) - 2\left(\boldsymbol{a}, \sum_{k=1}^{2} c_k\boldsymbol{u}_k\right) + \|\boldsymbol{a}\|^2\right\} \\ &= \frac{1}{2}\left\{\sum_{k,l=1}^{2} c_k c_l(\boldsymbol{u}_k, \boldsymbol{u}_l) - 2\sum_{k=1}^{2} c_k(\boldsymbol{a}, \boldsymbol{u}_k) + \|\boldsymbol{a}\|^2\right\} \end{aligned}$$

Partially differenciate this with recpect to c_i $(i = 1, 2)$.

$$\begin{aligned}
\frac{\partial J}{\partial c_i} &= \frac{\partial}{\partial c_i}\frac{1}{2}\left\{\sum_{k,l=1}^{2} c_k c_l(\boldsymbol{u}_k, \boldsymbol{u}_l) - 2\sum_{k=1}^{2} c_k(\boldsymbol{a}, \boldsymbol{u}_k) + \|\boldsymbol{a}\|^2\right\} \\
&= \frac{1}{2}\left\{\frac{\partial}{\partial c_i}\sum_{k,l=1}^{2} c_k c_l(\boldsymbol{u}_k, \boldsymbol{u}_l) - 2\frac{\partial}{\partial c_i}\sum_{k=1}^{2} c_k(\boldsymbol{a}, \boldsymbol{u}_k)\right\} \\
&= \frac{1}{2}\left\{2\sum_{k=1}^{2} c_k(\boldsymbol{u}_k, \boldsymbol{u}_i) - 2(\boldsymbol{a}, \boldsymbol{u}_i)\right\} \\
&= \sum_{k=1}^{2} c_k(\boldsymbol{u}_k, \boldsymbol{u}_i) - (\boldsymbol{a}, \boldsymbol{u}_i)
\end{aligned}$$

By writing $\dfrac{\partial J}{\partial c_1} = 0$ and $\dfrac{\partial J}{\partial c_2} = 0$ in matrix form, we obtain

$$\begin{pmatrix} (\boldsymbol{u}_1, \boldsymbol{u}_1) & (\boldsymbol{u}_2, \boldsymbol{u}_1) \\ (\boldsymbol{u}_1, \boldsymbol{u}_2) & (\boldsymbol{u}_2, \boldsymbol{u}_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} (\boldsymbol{a}, \boldsymbol{u}_1) \\ (\boldsymbol{a}, \boldsymbol{u}_2) \end{pmatrix}$$

By substituting column vectors $\boldsymbol{a}$, $\boldsymbol{u}_1$, and $\boldsymbol{u}_2$ in the above equation we obtain

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \end{pmatrix}$$

By solving this we obtain

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

Thus the linear combination of $\boldsymbol{u}_1$ and bmu_2 that is closest to the vector $\boldsymbol{a}$ is obtained as follows.

$$4\boldsymbol{u}_1 - \boldsymbol{u}_2 = 4\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$$

Solution 2 By substituting $\boldsymbol{a}$, $\boldsymbol{u}_1$, and $\boldsymbol{u}_2$ in J we obtain

$$\begin{aligned}
J &= \frac{1}{2}\|c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 - \boldsymbol{a}\|^2 \\
&= \frac{1}{2}\left\|c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}\right\|^2 \\
&= \frac{1}{2}\left\|\begin{pmatrix} c_1 + c_2 - 3 \\ c_1 - 2 \\ c_1 - 6 \end{pmatrix}\right\|^2 \\
&= \frac{1}{2}\left\{c_1^2 + c_2^2 + 9 + 2c_1c_2 - 6c_1 - 6c_2 + c_1^2 - 4c_1 + 4 + c_1^2 - 12c_1 + 36\right\} \\
&= \frac{1}{2}\left\{3c_1^2 + c_2^2 + 2c_1c_2 - 22c_1 - 6c_2 + 49\right\}
\end{aligned}$$

Partially differenciate this with respect to c_1 and c_2.

$$\begin{aligned}\frac{\partial J}{\partial c_1} &= \frac{1}{2}\{6c_1 + 2c_2 - 22\} = 3c_1 + c_2 - 11 \\ \frac{\partial J}{\partial c_2} &= \frac{1}{2}\{2c_1 + 2c_2 - 6\} = c_1 + c_2 - 3\end{aligned}$$

Then we obtain the following systems of equations.

$$\begin{aligned}3c_1 + c_2 &= 11 \\ c_1 + c_2 &= 3\end{aligned}$$

By solving this we obtain $c_1 = 4, c_2 = -1$. Thus the linear combination of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ that is closest to the vector $\boldsymbol{a}$ is obtained as follows.

$$4\boldsymbol{u}_1 - \boldsymbol{u}_2 = 4\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}3\\4\\4\end{pmatrix}$$

Problem 4 (10 points) Calculate the Fourier series expansion of the function $f(x) = x^2$ on the range $[-\pi, \pi]$ following the two steps (1) and (2).

(1) Calculate the linear combination of the following orthogonal functions that is closest to the function $f(x)$. As for the measure of the distance, use (the half of) the integral of the square of the difference on the range $[-\pi, \pi]$.

$$\{\frac{1}{2}, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx\}$$

(2) Show the Fourier series of $f(x)$ on the range $[-\pi, \pi]$. (The Fourier series of $f(x) = x^2$ is the limit of the linear combination obtained in (1) as n goes to infinity.)

Solution

(1) Assume the following equation holds. (Note: There are no coefficients a_0, …, a_n, b_1, …, b_n that satisfy the equation, but it's ok.)

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \tag{1}$$

Integrate the both sides of the equation (1) on the range $[-\pi, \pi]$.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\mathrm{d}x &= \int_{-\pi}^{\pi} \frac{1}{2}a_0\mathrm{d}x \\ &= a_0\pi\end{aligned}$$

Then we calculate a_0 as follows.

$$\begin{aligned}a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\mathrm{d}x \\ &= \frac{2}{3}\pi^2\end{aligned}$$

Multiply the both sides of the equation (1) by $\cos kx$ and integrate them on the range $[-\pi, \pi]$.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos kx\mathrm{d}x &= a_k \int_{-\pi}^{\pi} \cos^2 kx\mathrm{d}x \\ &= a_k\pi\end{aligned}$$

Then we calculate a_k as follows.

$$\begin{aligned}a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos kx\mathrm{d}x \\ &= \frac{1}{\pi}\left\{\left[x^2\frac{\sin kx}{k}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\frac{\sin kx}{k}\mathrm{d}x\right\} \\ &= -\frac{2}{\pi k}\int_{-\pi}^{\pi} x\sin kx\mathrm{d}x \qquad (\text{since } \left[x^2\frac{\sin kx}{k}\right]_{-\pi}^{\pi} \text{ is } 0)\end{aligned}$$

Here we calculate the integral $\int_{-\pi}^{\pi} x\sin kx\mathrm{d}x$.

$$\begin{aligned}\int_{-\pi}^{\pi} x\sin kx\mathrm{d}x &= \left[x\frac{-\cos kx}{k}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{-\cos kx}{k}\mathrm{d}x \\ &= -\frac{1}{k}\left[x\cos kx\right]_{-\pi}^{\pi} \qquad (\text{since } \int_{-\pi}^{\pi}\frac{-\cos kx}{k}\mathrm{d}x \text{ is } 0) \\ &= -\frac{1}{k}(\pi\cos\pi k - (-\pi)\cos(-\pi k)) \\ &= -\frac{1}{k}(\pi\cos\pi k + \pi\cos\pi k) \\ &= -\frac{2\pi}{k}\cos\pi k \\ &= -\frac{2\pi}{k}(-1)^k\end{aligned}$$

We resume the calculation of a_k.

$$\begin{aligned}a_k &= -\frac{2}{\pi k}(-\frac{2\pi}{k}(-1)^k) \\ &= \frac{4}{k^2}(-1)^k\end{aligned}$$

Multiply the both sides of the equation (1) by $\sin kx$ and integrate them on the range $[-\pi, \pi]$.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin kx\mathrm{d}x &= b_k \int_{-\pi}^{\pi} \sin^2 kx\mathrm{d}x \\ &= b_k\pi\end{aligned}$$

Then we calculate b_k as follows.

$$\begin{aligned}b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin kx\mathrm{d}x \\ &= 0 \quad (\text{since } x^2\sin kx \text{ is an odd function})\end{aligned}$$

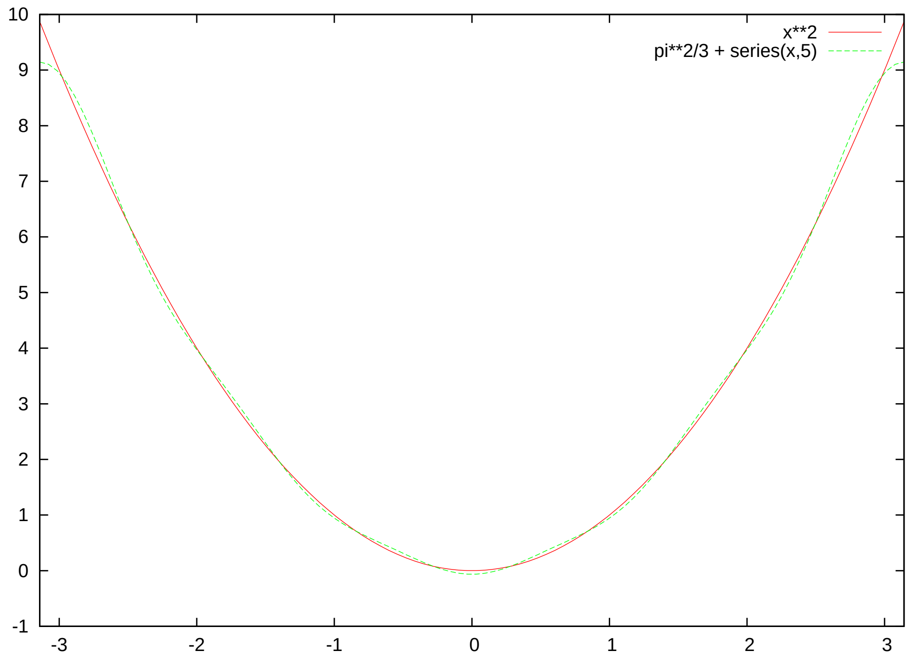

Figure 3: Comparison between the function $f(x) = x^2$ and the partial sum up to the term of $\cos 5x$

So the linear combination that is closest to the function $f(x)$ is

$$\frac{\pi^2}{3} + \sum_{k=1}^{n} \frac{4}{k^2}(-1)^k \cos kx.$$

(2) The Fourier expansion of $f(x) = x^2$ is the limit of the linear combination obtained in (1) as n goes to infinity:

$$\frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2}(-1)^k \cos kx.$$

Supplement: We depict the partial summation of this series to the term of $\cos 5x$

$$\frac{\pi^2}{3} + \sum_{k=1}^{5} \frac{4}{k^2}(-1)^k \cos kx = \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \frac{1}{4}\cos 4x - \frac{4}{25}\cos 5x$$

and $f(x) = x^2$ in Fig. 3.