A solution for Exercise 1

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Exercise 1 (Least square method)

Fit a straight line (a linear function) to the following three points so that (the half of) the sum of the squares of the distances of those points from the straight line is minimum, where the distance is measured in the vertical direction (the y-direction).

$$(0,1),(1,0),(2,-2)$$

A solution:

Let the line (the linear function) be $f(x) = ax+b$ and $(x_1, y_1) = (0, 1), (x_2, y_2) = (1, 0), (x_3, y_3) = (2, -2)$. The half of the sum of the squares of the distances of these points from the line is given as follows.

$$J = \frac{1}{2}\sum_{i=1}^{3}\{f(x_i) - y_i\}^2 = \frac{1}{2}\sum_{i=1}^{3}(ax_i + b - y_i)^2$$

J takes the minimum value in the point where the partial derivatives of J with respect to a and b are 0.

$$\frac{\partial J}{\partial a} = 0, \quad \frac{\partial J}{\partial b} = 0$$

Firstly the partial derivative of J with respect to a is calculated as follows.

$$\begin{aligned}
\frac{\partial J}{\partial a} &= \frac{\partial}{\partial a}\{\frac{1}{2}\sum_{i=1}^{3}(ax_i + b - y_i)^2\} \\
&= \frac{1}{2}\sum_{i=1}^{3}\frac{\partial}{\partial a}(ax_i + b - y_i)^2 \\
&= \frac{1}{2}\sum_{i=1}^{3}2(ax_i + b - y_i)x_i \\
&= \sum_{i=1}^{3}(ax_i + b - y_i)x_i \\
&= \sum_{i=1}^{3}(ax_i^2 + bx_i - x_iy_i) \\
&= a\sum_{i=1}^{3}x_i^2 + b\sum_{i=1}^{3}x_i - \sum_{i=1}^{3}x_iy_i
\end{aligned}$$

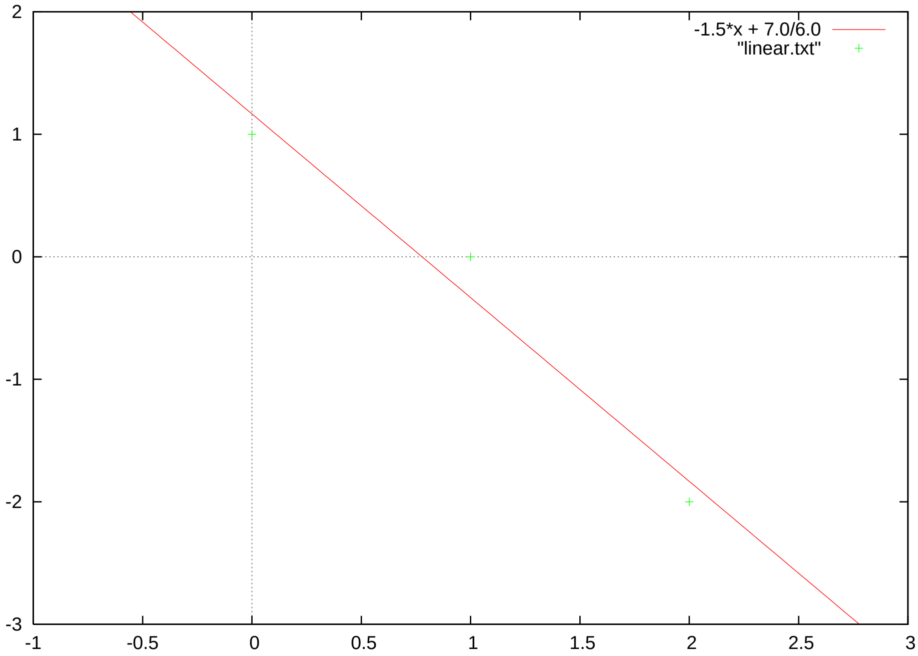

Figure 1: The straight line closest to the given three points

Secondly the partial derivative of J with respect to b is calculated as follows.

$$\begin{aligned}
\frac{\partial J}{\partial b} &= \frac{\partial}{\partial b}\{\frac{1}{2}\sum_{i=1}^{3}(ax_i + b - y_i)^2\} \\
&= \frac{1}{2}\sum_{i=1}^{3}\frac{\partial}{\partial b}(ax_i + b - y_i)^2 \\
&= \frac{1}{2}\sum_{i=1}^{3}2(ax_i + b - y_i) \\
&= \sum_{i=1}^{3}(ax_i + b - y_i) \\
&= a\sum_{i=1}^{3}x_i + b\sum_{i=1}^{3}1 - \sum_{i=1}^{3}y_i
\end{aligned}$$

Then we obtain the system of equations

$$\begin{aligned}
5a + 3b + 4 &= 0 \\
3a + 3b + 1 &= 0
\end{aligned}$$

and $a = -\frac{3}{2}, b = \frac{7}{6}$ is the solution. Hence the function is obtained as follows.

$$f(x) = -\frac{3}{2}x + \frac{7}{6}$$

The function is depicted with the three points in Fig. 1.