An introduction to Fourier transforms

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This document is largely based on the reference book [1] with some parts slightly changed.

1 Fourier Integral

Given a function $f(x)$, define $f_L(x)$ as follows.

$$f_L(x) = \begin{cases} f(x) & -L \leq x \leq L \\ 0 & \text{otherwise} \end{cases}$$

Note that

$$f(x) = \lim_{L \to \infty} f_L(x).$$

Also suppose $f_L(x)$ is represented by a Fourier series

$$f_L(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos \omega_k x + b_k \sin \omega_k x)$$

on the range $[-L, L]$, where a_0, a_k, b_k, ω_k are given as follows.

$$\begin{aligned} \omega_k &= \frac{k\pi}{L} \\ a_0 &= \frac{1}{L}\int_{-L}^{L} f_L(x)\mathrm{d}x \\ a_k &= \frac{1}{L}\int_{-L}^{L} f_L(x)\cos \omega_k x \mathrm{d}x \\ b_k &= \frac{1}{L}\int_{-L}^{L} f_L(x)\sin \omega_k x \mathrm{d}x \end{aligned}$$

Then $f_L(x)$ is represented as follows.

$$\begin{aligned} f_L(x) &= \frac{1}{2L}\int_{-L}^{L} f_L(x)\mathrm{d}x + \frac{1}{L}\sum_{k=1}^{\infty}\Big\{\cos\omega_k x\int_{-L}^{L} f_L(x)\cos\omega_k x\mathrm{d}x \\ &\qquad + \sin\omega_k x\int_{-L}^{L} f_L(x)\sin\omega_k x\mathrm{d}x\Big\} \end{aligned}$$

We now set

$$\Delta\omega = \omega_{k+1} - \omega_k = \frac{(k+1)\pi}{L} - \frac{k\pi}{L} = \frac{\pi}{L}.$$

Since $1/L = \Delta\omega/\pi$ we may rewrite the above equation as follows.

$$\begin{aligned} f_L(x) &= \frac{1}{2L}\int_{-L}^{L} f_L(x)\mathrm{d}x + \frac{\Delta\omega}{\pi}\sum_{k=1}^{\infty}\Big\{\cos\omega_k x\int_{-L}^{L} f_L(x)\cos\omega_k x\mathrm{d}x \\ &\qquad + \sin\omega_k x\int_{-L}^{L} f_L(x)\sin\omega_k x\mathrm{d}x\Big\} \\ &= \frac{1}{2L}\int_{-L}^{L} f_L(x)\mathrm{d}x + \frac{1}{\pi}\sum_{k=1}^{\infty}\Big\{\cos\omega_k x\int_{-L}^{L} f_L(x)\cos\omega_k x\mathrm{d}x \\ &\qquad + \sin\omega_k x\int_{-L}^{L} f_L(x)\sin\omega_k x\mathrm{d}x\Big\}\Delta\omega \\ &= \frac{1}{2L}\int_{-L}^{L} f_L(x)\mathrm{d}x + \frac{1}{\pi}\sum_{k=1}^{\infty}\Big\{\cos k\Delta\omega x\int_{-L}^{L} f_L(x)\cos k\Delta\omega x\mathrm{d}x \\ &\qquad + \sin k\Delta\omega x\int_{-L}^{L} f_L(x)\sin k\Delta\omega x\mathrm{d}x\Big\}\Delta\omega \end{aligned}$$

Now we would like to let $L \to \infty$, but before that we assume that $f(x)$ is *absolutely integrable* on the x-axis; that is, the following finite limits exist.

$$\lim_{a\to-\infty}\int_{a}^{0}|f(x)|\mathrm{d}x + \lim_{a\to\infty}\int_{0}^{a}|f(x)|\mathrm{d}x$$

Note that this is written as $\displaystyle\int_{-\infty}^{\infty}|f(x)|\mathrm{d}x$.

We now let $L \to \infty$, then the first term $\displaystyle\frac{1}{2L}\int_{-L}^{L} f_L(x)\mathrm{d}x$ approaches zero. Also $\Delta\omega = \pi/L$ approaches zero and it seems *plausible* that the infinite series becomes an integral from 0 to ∞ as follows[1][2].

$$f(x) = \frac{1}{\pi}\int_0^\infty \left\{\cos\omega x \int_{-\infty}^\infty f(x)\cos\omega x \mathrm{d}x + \sin\omega x \int_{-\infty}^\infty f(x)\cos\omega x \mathrm{d}x\right\}\mathrm{d}\omega$$

By introducing the functions

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(x)\cos\omega x \mathrm{d}x \tag{1}$$

and

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(x)\sin\omega x \mathrm{d}x \tag{2}$$

we can also write the above formula as follows.

$$f(x) = \int_0^\infty \{A(\omega)\cos\omega x + B(\omega)\sin\omega x\}\,\mathrm{d}\omega \tag{3}$$

This is called a representation of $f(x)$ by a **Fourier integral**.

The following theorem, which we have already seen in example14-1.pdf, holds (see p. 513 of the reference book [1]).

Theorem 1 *If $f(x)$ is piecewise continuous in every finite interval and has a right-hand derivative and a left-hand derivative at every point and if $f(x)$ is absolutely integrable, then $f(x)$ can be represented by (3) with A and B given by (1) and (2). At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point. In formula,*

$$\int_0^\infty \{A(\omega)\cos\omega x + B(\omega)\sin\omega x\}\,\mathrm{d}\omega = \frac{f(x-0)+f(x+0)}{2}.$$

□

[1] It just seems plausible and is not the logical consequence, although the equality actually holds similarly to the previous example we have seen in example14-1.pdf, provided that $f(x)$ is piecewise continuous in every finite interval and has a right-hand derivative and a left-hand derivative at every point. In textbooks used in engineering classes, Fourier transform is usually explained as the limit of Fourier series as the period L goes to infinity.

[2] Notice that I use the same name x for a few variables each of which has different scope. Of course we can rename inner x as other name such as v, which might be a usual way of writing this kind of formula.

Example Calculate the Fourier integral of the following function.

$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution We calculate $A(\omega)$ and $B(\omega)$ as follows.

$$\begin{aligned} A(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x \mathrm{d}x \\ &= \frac{1}{\pi}\int_{-1}^{1} \cos\omega x \mathrm{d}x \\ &= \frac{1}{\pi}\left[\frac{\sin\omega x}{\omega}\right]_{-1}^{1} \\ &= \frac{2\sin\omega}{\pi\omega} \end{aligned}$$

$$\begin{aligned} B(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x \mathrm{d}x \\ &= \frac{1}{\pi}\int_{-1}^{1} \sin\omega x \mathrm{d}x \\ &= 0 \end{aligned}$$

So we obtain the Fourier integral of $f(x)$ as follows.

$$\frac{2}{\pi}\int_{0}^{\infty} \frac{\cos\omega x\sin\omega}{\omega}\mathrm{d}\omega$$

By Theorem 1 we obtain the following equality.

$$\frac{2}{\pi}\int_{0}^{\infty} \frac{\cos\omega x\sin\omega}{\omega}\mathrm{d}\omega = \begin{cases} 1 & -1 < x < 1 \\ 1/2 & x = -1, 1 \\ 0 & \text{otherwise} \end{cases}$$

We should mention one thing. The integral above is the limit of the formula

$$\frac{2}{\pi}\int_{0}^{a} \frac{\cos\omega x\sin\omega}{\omega}\mathrm{d}\omega$$

as a goes to infinity. In this integral, there are oscillations near the points $x = -1$ and $x = 1$. The oscillations does not disappear even if a increases, similarly to the Fourier series. This is called the Gibbs phenomenon.

Note By setting $x = 0$ in the Fourier integral of $f(x)$, we obtain the following equality.

$$\frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega}{\omega}\mathrm{d}\omega = 1$$

By multiplying the both sides by $\frac{\pi}{2}$ we obtain the following equality.

$$\int_0^{\infty}\frac{\sin\omega}{\omega}\mathrm{d}\omega = \frac{\pi}{2}$$

This is called the **Dirichlet integral**. Consider the following (partial) integral (∞ is replaced by a) so called **sine integral**.

$$\mathrm{Si}(a) = \int_0^{a}\frac{\sin\omega}{\omega}\mathrm{d}\omega$$

In the sine integral $\mathrm{Si}(a)$, there are oscillations. The oscillations in the above integral come from the oscillations in the sine integral.

2 Fourier transform

The (real) Fourier integral is

$$f(x) = \int_0^{\infty}\{A(\omega)\cos\omega x + B(\omega)\sin\omega x\}\,\mathrm{d}\omega$$

where A and B are given as follows.

$$\begin{aligned} A(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\mathrm{d}v \\ B(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\mathrm{d}v \end{aligned}$$

Substituting A and B into the integral, we have

$$\begin{aligned}
f(x) &= \int_0^\infty \left\{ \frac{1}{\pi}\int_{-\infty}^\infty f(v)\cos\omega v \mathrm{d}v \cos\omega x + \frac{1}{\pi}\int_{-\infty}^\infty f(v)\sin\omega v \mathrm{d}v \sin\omega x \right\} \mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^\infty \left\{ \int_{-\infty}^\infty f(v)\cos\omega v \mathrm{d}v \cos\omega x + \int_{-\infty}^\infty f(v)\sin\omega v \mathrm{d}v \sin\omega x \right\} \mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^\infty \left\{ \int_{-\infty}^\infty f(v)\cos\omega v \cos\omega x \mathrm{d}v + \int_{-\infty}^\infty f(v)\sin\omega v \sin\omega x \mathrm{d}v \right\} \mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^\infty \left\{ \int_{-\infty}^\infty f(v)(\cos\omega v \cos\omega x + \sin\omega v \sin\omega x) \mathrm{d}v \right\} \mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^\infty \left\{ \int_{-\infty}^\infty f(v)\cos(\omega x - \omega v) \mathrm{d}v \right\} \mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^\infty \left\{ \int_{-\infty}^\infty f(v)\cos(\omega(x - v)) \mathrm{d}v \right\} \mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \left\{ \int_{-\infty}^\infty f(v)\cos(\omega(x - v)) \mathrm{d}v \right\} \mathrm{d}\omega
\end{aligned}$$

(since the integral in the brackets is an even function of ω)

The integral of this form with sin instead of cos

$$\frac{1}{2\pi}\int_{-\infty}^\infty \left\{ \int_{-\infty}^\infty f(v)\sin(\omega(x - v)) \mathrm{d}v \right\} \mathrm{d}\omega$$

is zero since the integral in the brackets is an odd function of ω.

By the Euler formula

$$e^{ix} = \cos x + i\sin x$$

we obtain the following equality.

$$e^{i\omega(x-v)} = \cos(\omega(x-v)) + i\sin(\omega(x-v))$$

By multiplying the both sides by $f(v)$ we obtain the following equality.

$$f(v)e^{i\omega(x-v)} = f(v)\cos(\omega(x-v)) + if(v)\sin(\omega(x-v))$$

By taking integral with respect to v and ω and multiplying the result by $\dfrac{1}{2\pi}$

we obtain the following equality.

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)e^{i\omega(x-v)}\mathrm{d}v\right\}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)\cos(\omega(x-v)) + if(v)\sin(\omega(x-v))\mathrm{d}v\right\}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)\cos(\omega(x-v))\mathrm{d}v\right\}\mathrm{d}\omega \\
&\quad +i\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)\sin(\omega(x-v))\mathrm{d}v\right\}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)\cos(\omega(x-v))\mathrm{d}v\right\}\mathrm{d}\omega \\
&= f(x)
\end{aligned}$$

So we obtain the following equality.

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)e^{i\omega(x-v)}\mathrm{d}v\right\}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)e^{i\omega x-i\omega v}\mathrm{d}v\right\}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)e^{i\omega x}e^{-i\omega v}\mathrm{d}v\right\}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)e^{-i\omega v}\mathrm{d}v\right\}e^{i\omega x}\mathrm{d}\omega
\end{aligned}$$

We usually write this as follows[3].

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\mathrm{d}\omega \\
F(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\mathrm{d}x
\end{aligned}$$

We call $\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\mathrm{d}x$ the Fourier transform of $f(x)$ and we call $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\mathrm{d}\omega$ the inverse Fourier transform of $F(\omega)$. Note that the

[3]The constants ($\frac{1}{2\pi}$ and 1) depend on textbooks. They may be $\frac{1}{\sqrt{2\pi}}$ and $\frac{1}{\sqrt{2\pi}}$.

above two equalities are just definitions of transformations and $f(x)$ might not be equal to the inverse Fourier transform of the Fourier transform of $f(x)$ (see Theorem 1).

Example Calculate the Fourier transform of the following function.

$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution

$$\begin{aligned} F(\omega) &= \int_{-1}^{1} e^{-i\omega x} \mathrm{d}x \\ &= \left[\frac{e^{-i\omega x}}{-i\omega} \right]_{-1}^{1} \\ &= \frac{1}{-i\omega}(e^{-i\omega} - e^{i\omega}) \\ &= \frac{1}{-i\omega}(\cos\omega - i\sin\omega - (\cos\omega + i\sin\omega)) \\ &= \frac{1}{-i\omega}(-2i\sin\omega) \\ &= 2\frac{\sin\omega}{\omega} \end{aligned}$$

References

[1] Erwin Kreyszig. *Advanced Engineering Mathematics.* John Wiley & Sons Ltd., tenth edition, 2011.