Example 14-1

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Recall the periodic rectangular function $f_L(x)$, which appeared in example14.pdf, of period $2L > 2$, which was defined as follows.

$$f_L(x) = \begin{cases} 0 & -L \le x \le -1 \\ 1 & -1 < x < 1 \\ 0 & 1 \le x \le L \end{cases}$$

The Fourier series of $f_L(x)$ was obtained as follows (see example14.pdf).

$$\frac{1}{L} + \sum_{k=1}^{\infty} \left\{ \frac{2}{L} \cdot \frac{\sin \omega_k}{\omega_k} \cos \omega_k x \right\}$$

where $\omega_k = \frac{k\pi}{L}$. Let's consider what happens when L goes to positive infinity. Let $\Delta\omega = \frac{\pi}{L}$.

$$\begin{aligned} & \lim_{L\to\infty} \left\{ \frac{1}{L} + \sum_{k=1}^{\infty} \left\{ \frac{2}{L} \cdot \frac{\sin \omega_k}{\omega_k} \cos \omega_k x \right\} \right\} \\ = & \lim_{L\to\infty} \sum_{k=1}^{\infty} \left\{ \frac{2}{L} \cdot \frac{\sin \omega_k}{\omega_k} \cos \omega_k x \right\} \\ = & \lim_{L\to\infty} \sum_{k=1}^{\infty} \left\{ \frac{2\Delta\omega}{\pi} \cdot \frac{\sin k\Delta\omega}{k\Delta\omega} \cos k\Delta\omega x \right\} \\ = & \frac{2}{\pi} \lim_{L\to\infty} \sum_{k=1}^{\infty} \left\{ \Delta\omega \cdot \frac{\sin k\Delta\omega}{k\Delta\omega} \cos k\Delta\omega x \right\} \end{aligned}$$

It *seems plausible* that the above formula is equal to the following formula.

$$\frac{2}{\pi} \int_0^{\infty} \frac{\sin \omega}{\omega} \cos \omega x \mathrm{d}\omega$$

Note that these two formulas are actually equal but the equality is not the logical consequence of the above formula and the definition of integral (such as the Riemann integral), since the notation

$$\int_0^\infty \cdots$$

is an abbreviation for

$$\lim_{a\to\infty}\int_0^a \cdots$$

but the previous formula, which has two limit operators (with one of them being abbreviated in the summation), does not match this form.

The equality above is proved as follows by using the following Theorem (see p. 513 of the reference book [1]).

Theorem 1 *If $f(x)$ is piecewise continuous in every finite interval and has a right-hand derivative and a left-hand derivative at every point and if $f(x)$ is absolutely integrable, then the following equation holds.*

$$\int_0^\infty \{A(\omega)\cos\omega x + B(\omega)\sin\omega x\}\,\mathrm{d}\omega = \frac{f(x-0)+f(x+0)}{2}$$

where the functions A and B are defined as follows.

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\mathrm{d}x$$
$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\mathrm{d}x$$

Note that $f(x-0)$ and $f(x+0)$ represent the left- and right-hand limits of $f(x)$. □

Firstly, by the property of Fourier series on the discontinuous points (please refer to Theorem 2 in p.480 in the reference book [1]), the following equality holds.

$$\lim_{L\to\infty}\left\{\frac{1}{L}+\sum_{k=1}^{\infty}\left\{\frac{2}{L}\cdot\frac{\sin\omega_k}{\omega_k}\cos\omega_k x\right\}\right\} = \begin{cases} 1 & -1 < x < 1 \\ 1/2 & x = -1, 1 \\ 0 & \text{otherwise} \end{cases}$$

Secondly, let's consider the following function f.

$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

This function f satisfies the conditions in Theorem 1. So, by applying Theorem 1, we obtain the following equality.

$$\int_0^\infty \{A(\omega)\cos\omega x + B(\omega)\sin\omega x\}\,\mathrm{d}\omega = \frac{f(x-0)+f(x+0)}{2}$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x \mathrm{d}x$$
$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x \mathrm{d}x$$

$A(\omega)$ and $B(\omega)$ is calculated as follows.

$$\begin{aligned} A(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x \mathrm{d}x \\ &= \frac{1}{\pi}\int_{-1}^{1} \cos\omega x \mathrm{d}x \\ &= \frac{1}{\pi}\left[\frac{\sin\omega x}{\omega}\right]_{-1}^{1} \\ &= \frac{1}{\pi\omega}(\sin\omega - \sin(-\omega)) \\ &= \frac{2}{\pi}\frac{\sin\omega}{\omega} \\ B(\omega) &= 0 \end{aligned}$$

So we obtain

$$\begin{aligned} \int_0^\infty \frac{2}{\pi}\frac{\sin\omega}{\omega}\cos\omega x \mathrm{d}\omega &= \frac{f(x-0)+f(x+0)}{2} \\ &= \begin{cases} 1 & -1 < x < 1 \\ 1/2 & x = -1, 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Hence the equality

$$\lim_{L\to\infty}\left\{\frac{1}{L}+\sum_{k=1}^{\infty}\left\{\frac{2}{L}\cdot\frac{\sin\omega_k}{\omega_k}\cos\omega_k x\right\}\right\}=\frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega}{\omega}\cos\omega x \mathrm{d}\omega$$

holds.

By the way, the following equation holds.

$$f(x)=\lim_{L\to\infty}f_L(x)$$

Note that this function f is a non-periodic function while $f_L(x)$ is a periodic one. The function

$$\frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega}{\omega}\cos\omega x \mathrm{d}\omega,$$

which appered above, is called the Fourier integral of the function f. We will argue the Fourier integral in ft.pdf.

References

[1] Erwin Kreyszig. *Advanced Engineering Mathematics*. John Wiley & Sons Ltd., tenth edition, 2011.